Groups of units of integral group rings commensurable with direct products of free-by-free groups *

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Abstract

We classify the finite groups G such that the group of units of the integral group ring $\mathbb{Z}G$ has a subgroup of finite index which is a direct product of free-by-free groups.

The investigations on the unit group $\mathbb{Z}G^*$ of the integral group ring $\mathbb{Z}G$ of a finite group G have a long history and go back to work of Higman [11]. One of the fundamental problems that attracts a lot of attention is Research Problem 17 posed by Sehgal in [30]: find presentations of $\mathbb{Z}G^*$ for some finite groups G. For many finite groups G a finite set of generic generators of a subgroup of finite index in $\mathbb{Z}G^*$ has been obtained, but there is no general result known on determining the relations among these generators. This work was initiated by Bass and Milnor [2] and then continued by Kleinert [18], Ritter and Sehgal [28], and Jespers and Leal [14]. For a survey on the above mentioned results, we refer to Sehgal's book [30] and to [13].

An alternative approach to that of finding presentations is the one suggested by Kleinert in [19]. Recall that a generic unit group of A is a subgroup of finite index in the group of reduced norm 1 elements of an order in A. Then according to Kleinert "a unit theorem for a finite dimensional simple rational algebra A consists of the definition, in purely group theoretical terms, of a class of groups $\mathcal{C}(A)$ such that almost all generic unit groups of A are members of $\mathcal{C}(A)$". This approach has an obvious generalization to finite dimensional semi-simple rational algebras, such as the rational group algebra $\mathbb{Q}G$ of a finite group G and its orders, for example $\mathbb{Z}G$. This kind of unit theorem has been obtained for integral group rings $\mathbb{Z}G$ of some restricted classes of finite groups G. We give a brief history on the descriptions obtained so far. Higman in [11] showed that if G is a finite abelian group then $\mathbb{Z}G^* = L \times (\pm G)$, where L is a free abelian group of rank depending on the cardinality of G and the order of the elements of G. This result heavily depends on Dirichlet's Unit Theorem. He also showed that if G is non-abelian then $\mathbb{Z}G^*$ is finite if and only if G is a Hamiltonian 2-group and in this case $\mathbb{Z}G^* = \pm G$. The finite groups G such that $\mathbb{Z}G^*$ is virtually free and non-abelian (there are only four) were classified in [12]. This last result was motivated by a previous theorem of Hartley and Pickel [10] which states that $\mathbb{Z}G^*$ is either abelian, finite or has a non-abelian free subgroup. Finally, the finite groups G such that $\mathbb{Z}G^*$ is virtually a direct product of free groups (there are infinitely many) were classified in a series of papers by Jespers, Leal and del Río [16, 17, 21]. Thus the finite groups G for which a unit theorem, in the sense of Kleinert, is known for $\mathbb{Z}G^*$ are those for which the class of groups considered are either finite groups, abelian groups, free groups or direct products of free groups. As far as we know, all the finite groups G for which the structure of $\mathbb{Z}G^*$ is known up to commensurability are covered by these results.

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The aim of this paper is to obtain a group theoretical description of $\mathbb{Z}G^*$ for a larger family of finite groups G than the family of groups mentioned in the previous paragraph. We do this by connecting the study of $\mathbb{Z}G^*$ with the better known structure of the Bianchi groups. The inspiration came from some examples in [5] and [26]. In the second reference some presentations of $\mathbb{Z}G^*$ are obtained for two groups of order 16 for which $\mathbb{Z}G^*$ is commensurable with the Picard group $\mathrm{PSL}_2(\mathbb{Z}[i])$. These two groups belong to a class of finite groups, called groups of Kleinian type, for which geometrical methods are applicable to obtain presentations of groups of finite index (implementation of the method however is usually difficult). Our main theorem (Theorem 1) shows that the class $\mathcal{C}$ containing the generic groups of $\mathbb{Z}G^*$ for G of Kleinian type is formed by the direct products of free-by-free groups, and in fact this property characterizes the groups of Kleinian type. Furthermore, we classify the finite groups of Kleinian type as the groups which are epimorphic images of some specific groups. This classification is the most involving part of the paper. In order to state this result we first fix some terminology.

Recall that a group H is said to be *free-by-free* if H contains a normal subgroup N so that both N and H/N are free groups. Note that the trivial and infinite cyclic group are free groups, and thus free groups and finitely generated abelian groups are direct products of free-by-free groups.

For a ring R we denote by R^* the group of invertible elements of R and by $Z(R)$ its centre. In case R is an order in a simple finite dimensional rational algebra A we denote by R^1 the group consisting of the elements of reduced norm 1 in R. (By an order we always mean a $\mathbb{Z}$-order; see [30] for a definition.)

Two subgroups H_1 and H_2 of a group H are said to be *commensurable* when their intersection has finite index in both H_1 and H_2. Often the group H is clear from the context and hence will not be specifically mentioned. For instance, the statement "$\mathbb{Z}G^*$ is commensurable with a direct product of free-by-free groups" means that $\mathbb{Z}G^*$ is commensurable with some subgroup of $\mathbb{Q}G^*$ which decomposes in a direct product of free-by-free groups. Similarly, if R is an order in a simple finite dimensional rational algebra A, then the statement "R^1 is commensurable with a free-by-free group" means that R^1 is commensurable with a subgroup of A^* with the mentioned property.

A finite group G is said to be of *Kleinian type* if every non-commutative simple quotient A of the rational group algebra $\mathbb{Q}G$ has an embedding $\psi : A \to M_2(\mathbb{C})$ such that $\psi(R^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ for some (every) order R in A.

It turns out that if G is a finite group of Kleinian type then it is metabelian. Hence to state our classification of the groups of Kleinian type it is convenient to introduce some notation for presentations of such groups. The cyclic group of order n is usually denoted by C_n. To emphasize that $a \in C_n$ is a generator of the group, we write C_n either as $\langle a\rangle$ or $\langle a\rangle_n$. Recall that a group G is metabelian if G has an abelian normal subgroup N such that $A = G/N$ is abelian. We simply denote this information as $G = N : A$. To give a concrete presentation of G we will write N and A as direct products of cyclic groups and give the necessary extra information on the relations between these generators. By $\overline{x}$ we denote the coset xN. For example, the dihedral group of order $2n$ and the quaternion group of order $4n$ can be described as

$$\begin{aligned} D_{2n} &= \langle a\rangle_n : \langle\overline{b}\rangle_2, \quad b^2 = 1, a^b = a^{-1}. \\ Q_{4n} &= \langle a\rangle_{2n} : \langle\overline{b}\rangle_2, \quad a^b = a^{-1},\ b^2 = a^n. \end{aligned}$$

If N has a complement in G then A can be identify with this complement and we write $G = N \rtimes A$. For example, the dihedral group also can be given by $D_{2n} = \langle a\rangle_n \rtimes \langle b\rangle_2$ with $a^b = a^{-1}$.

We are now in a position to formulate the main result.

Theorem 1 *For a finite group G the following statements are equivalent.*

(A) $\mathbb{Z}G^*$ *is commensurable with a direct product of free-by-free groups.*

(B) For every simple quotient A of $\mathbb{Q}G$ and some (every) order R in A, R^1 is commensurable with a free-by-free group.

(C) For every simple quotient A of $\mathbb{Q}G$ and some (every) order R in A, R^1 has virtual cohomological dimension at most 2.

(D) G is of Kleinian type.

(E) Every simple quotient of $\mathbb{Q}G$ is either a field, a totally definite quaternion algebra or $M_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$.

(F) G is either abelian or an epimorphic image of $A \times H$, where A is abelian and one of the following conditions holds:

1. *A has exponent 6 and H is one of the following groups:*
   - $\mathcal{W} = \left(\langle t\rangle_2 \times \langle x^2\rangle_2 \times \langle y^2\rangle_2\right) : (\langle \overline{x}\rangle_2 \times \langle \overline{y}\rangle_2)$, *with* $t = (y, x)$ *and* $Z(\mathcal{W}) = \langle x^2, y^2, t\rangle$.
   - $\mathcal{W}_{1n} = \left(\prod_{i=1}^{n} \langle t_i\rangle_2 \times \prod_{i=1}^{n} \langle y_i\rangle_2\right) \rtimes \langle x\rangle_4$, *with* $t_i = (y_i, x)$ *and* $Z(\mathcal{W}_{1n}) = \langle t_1, \dots, t_n, x^2\rangle$.
   - $\mathcal{W}_{2n} = \left(\prod_{i=1}^{n} \langle y_i\rangle_4\right) \rtimes \langle x\rangle_4$, *with* $t_i = (y_i, x) = y_i^2$ *and* $Z(\mathcal{W}_{2n}) = \langle t_1, \dots, t_n, x^2\rangle$.
2. *A has exponent 4 and H is one of the following groups:*
   - $\mathcal{V} = \left(\langle t\rangle_2 \times \langle x^2\rangle_4 \times \langle y^2\rangle_4\right) : (\langle \overline{x}\rangle_2 \times \langle \overline{y}\rangle_2)$, *with* $t = (y, x)$ *and* $Z(\mathcal{W}) = \langle x^2, y^2, t\rangle$.
   - $\mathcal{V}_{1n} = \left(\prod_{i=1}^{n} \langle t_i\rangle_2 \times \prod_{i=1}^{n} \langle y_i\rangle_4\right) \rtimes \langle x\rangle_8$, *with* $t_i = (y_i, x)$ *and* $Z(\mathcal{V}_{1n}) = \langle t_1, \dots, t_n, y_1^2, \dots, y_n, x^2\rangle$.
   - $\mathcal{V}_{2n} = \left(\prod_{i=1}^{n} \langle y_i\rangle_8\right) \rtimes \langle x\rangle_8$, *with* $t_i = (y_i, x) = y_i^4$ *and* $Z(\mathcal{V}_{2n}) = \langle t_i, x^2\rangle$.
   - $\mathcal{U}_1 = \left(\prod_{1\le i<j\le 3} \langle t_{ij}\rangle_2 \times \prod_{k=1}^{3} \langle y_k^2\rangle_2\right) : \left(\prod_{k=1}^{3} \langle \overline{y_k}\rangle_2\right)$, *with* $t_{ij} = (y_j, y_i)$ *and* $Z(\mathcal{U}_1) = \langle t_{12}, t_{13}, t_{23}, y_1^2, y_2^2, y_3^2\rangle$
   - $\mathcal{U}_2 = \left(\langle t_{23}\rangle_2 \times \langle y_1^2\rangle_2 \times \langle y_2^2\rangle_4 \times \langle y_3^2\rangle_4\right) : \left(\prod_{k=1}^{3} \langle \overline{y_k}\rangle_2\right)$, *with* $t_{ij} = (y_j, y_i)$, $y_2^4 = t_{12}$, $y_3^4 = t_{13}$ *and* $Z(\mathcal{U}_2) = \langle t_{12}, t_{13}, t_{23}, y_1^2, y_2^2, y_3^2\rangle$.
3. *A has exponent 2 and H is one of the following groups:*
   - $\mathcal{T} = (\langle t\rangle_4 \times \langle y\rangle_8) : \langle \overline{x}\rangle_2$, *with* $t = (y, x)$ *and* $x^2 = t^2 = (x, t)$.
   - $\mathcal{T}_{1n} = \left(\prod_{i=1}^{n} \langle t_i\rangle_4 \times \prod_{i=1}^{n} \langle y_i\rangle_4\right) \rtimes \langle x\rangle_8$, *with* $t_i = (y_i, x)$, $(t_i, x) = t_i^2$ *and* $Z(\mathcal{T}_{1n}) = \langle t_1^2, \dots, t_n^2, x^2\rangle$.
   - $\mathcal{T}_{2n} = \left(\prod_{i=1}^{n} \langle y_i\rangle_8\right) \rtimes \langle x\rangle_4$, *with* $t_i = (y_i, x) = y_i^{-2}$ *and* $Z(\mathcal{T}_{2n}) = \langle t_1^2, \dots, t_n^2, x^2\rangle$.
   - $\mathcal{T}_{3n} = \left(\prod_{i=2}^{n} \langle t_i\rangle_4 \times \langle y_1^2 t_1\rangle_2 \times \langle y_1\rangle_8 \times \prod_{i=2}^{n} \langle y_i\rangle_4\right) : \langle \overline{x}\rangle_2$, *with* $t_i = (y_i, x)$, $(t_i, x) = t_i^2$, $x^2 = t_1^2$, *and* $Z(\mathcal{T}_{3n}) = \langle t_1^2, \dots, t_n^2, x^2\rangle$.
4. *$H = M \rtimes P = (M \times Q) : \langle \overline{u}\rangle_2$, where M is an elementary abelian 3-group, $P = Q : \langle \overline{u}\rangle_2$, $m^u = m^{-1}$ for every $m \in M$, and one of the following conditions holds:*
   - *A has exponent 4 and $P = C_8$.*
   - *A has exponent 6, $P = \mathcal{W}_{1n}$ and $Q = \langle y_1, \dots, y_n, t_1, \dots, t_n, x^2\rangle$.*
   - *A has exponent 2, $P = \mathcal{W}_{21}$ and $Q = \langle y_1^2, x\rangle$.*

According to [31], a group G is called *good* if the homomorphism of cohomology groups $H^n(\widehat{G}, M) \longrightarrow H^n(G, M)$ induced by the natural homomorphism $G \longrightarrow \widehat{G}$ of G to its profinite completion $\widehat{G}$ is an isomorphism for every finite G-module M.

Theorem 1 yields that for a finite group G of Kleinian type the non-commutative simple components of $\mathbb{Q}G$ that are not totally definite quaternion algebras are of the form $M_2(\mathbb{Q}(\sqrt{-d}))$ with $d = 0, 1, 2$ or 3. On the other hand the groups of units of an order in a number field and in a totally definite quaternion algebra are commensurable with a free abelian group. Therefore, since the group of units of two orders in $\mathbb{Q}G$ are commensurable, Theorem 1 implies that $\mathbb{Z}G^*$ is commensurable with a direct product of a free abelian group and groups of the form $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ with $d = 0, 1, 2$ or 3. Recently it was shown that $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-d}])$ is good for every non-negative integer [9]. Since the class of good groups is closed under commensurability and $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{-d}])$ has a subgroup of finite index isomorphic to a subgroup of finite index of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$, it follows that $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-d}])$ is good. Moreover the class of good groups is closed under finite direct products. Hence the following property follows at once.

Corollary 2 *If G is a finite group of Kleinian type then the group of units of its integral group ring $\mathbb{Z}G$ is good.*

In particular, this corollary says that the virtual cohomological dimension of the profinite completion of ZG^* coincides with the virtual cohomological dimension of ZG^* and so the profinite completion of ZG^* is virtually torsion free.

The outline of the paper is as follows. In Section 1 we introduce the basic notation used throughout the paper. In Section 2 we show that conditions (A) and (B) are equivalent. In Section 3 we prove (B) implies (C) (which is obvious), (C) implies (D) (by first classifying the simple algebras of Kleinian type and the finite dimensional simple algebras A for which R^1 has virtual cohomological dimension at most 2 for an order R in A) and (E) implies (B) (by using known facts about Euclidean Bianchi groups). Section 4 is dedicated to prove (F) implies (E). At this point one has shown that all the groups satisfying condition (F) are of Kleinian type. The most involved part of the proof is to show that (D) implies (F), that is showing that condition (F) exhausts the class of groups of Kleinian type. This is proved for nilpotent groups in Section 5 and for non-nilpotent groups in Section 6.

In a preliminary version of the proof of (D) implies (F) we used previous results from [17, 26]. We thank Jairo Gonçalves for attracting our attention to an old result of Amitsur which classifies the finite groups that have all its irreducible complex characters of degree 1 or 2. This result has been very helpful in reducing earlier given arguments and in making the proof of (D) implies (F) independent of [17, 26].

1 Preliminaries

We introduce the basic notation and the main tools used in the paper. The Euler function is denoted by φ. For a positive integer n, let ξ_n denote a complex primitive root of unity.

Let G be a group. For $x, y \in G$, we put $x^y = y^{-1}xy$ and $(x, y) = xyx^{-1}y^{-1}$. We recall the following well known formulas: $(ab, c) = (b, c)^{a^{-1}}(a, c)$ and $(a, bc) = (a, b)(a, c)^{b^{-1}}$. The centre and derived subgroup of G are denoted by $Z(G)$ and G' respectively. The notation $H \leq G$ means that H is a subgroup of G and if H is a normal subgroup of G then we write $H \trianglelefteq G$. The normalizer of $H \leq G$ in G is denoted by $N_G(H)$. If $N \trianglelefteq G$ then we will use the usual bar notation for the natural

images of the elements and subsets of G in G/N, that is $\overline{x}$ denotes the coset xN of $x \in G$ and if $X \subseteq G$ then $\overline{X}$ denotes $\{\overline{x} \mid x \in X\}$. A semidirect product associated to an action of a group H on a group N is denoted by $N \rtimes H$.

We say that a group *virtually satisfies* a group theoretical condition if it has a subgroup of finite index satisfying the given condition. For example, G is virtually abelian if and only if G has an abelian subgroup of finite index. Notice that if a class of groups satisfying a property $\mathcal{P}$ is closed under subgroups of finite index then a group G is commensurable with a group satisfying $\mathcal{P}$ if and only if it virtually satisfies $\mathcal{P}$. Moreover, in this case, G is commensurable with a group which is a direct product of groups satisfying $\mathcal{P}$ if and only if it is virtually a direct product of groups satisfying $\mathcal{P}$. This, of course, applies to the class of free-by-free groups.

As well as the groups described in statement (F) of Theorem 1, the following metabelian groups will be relevant.

$$\begin{array}{lcll} D^+_{2^{n+2}} & = & \langle a\rangle_{2^{n+1}} \rtimes \langle b\rangle_2, & a^b = a^{2^n+1}. \\ D^-_{2^{n+2}} & = & \langle a\rangle_{2^{n+1}} \rtimes \langle b\rangle_2, & a^b = a^{2^n-1}. \\ \mathcal{D} & = & (\langle c\rangle_4 \times \langle a\rangle_2) \rtimes \langle b\rangle_2, & Z(\mathcal{D}) = \langle c\rangle,\ (b,a) = c^2. \\ \mathcal{D}^+ & = & (\langle c\rangle_4 \times \langle a\rangle_4) \rtimes \langle b\rangle_2, & Z(\mathcal{D}^+) = \langle c\rangle,\ (b,a) = ca^2. \end{array}$$

Recall that if G is a non-abelian group of order 2^{n+2} having a cyclic subgroup of index 2 then G is isomorphic to either the dihedral group $D_{2^{n+2}}$, the quaternion group $Q_{2^{n+2}}$ or one of the two semi-dihedral groups $D^+_{2^{n+2}}$ or $D^-_{2^{n+2}}$.

If K is a field and a and b are two non zero elements of K then $\left(\frac{a,b}{K}\right)$ denotes the quaternion K-algebra defined by a and b, that is, the K-algebra given by the following presentation:

$$\left(\frac{a,b}{K}\right) = K[i,j|i^2 = a, j^2 = b, ji = -ij].$$

In case $a = b = -1$, then the previous algebra is also denoted $\mathbb{H}(K)$. It is well known that $\left(\frac{a,b}{K}\right)$ is split, that is, it is isomorphic to $M_2(K)$, if and only if the equation $aX^2 + bY^2 = Z^2$ has a solution different from $X = Y = Z = 0$.

Let A be a finite dimensional semi-simple rational algebra and R an order in A. Then R^* is commensurable with the group of units of every order in A (see for example [30, Lemma 4.6]). Assume that, furthermore, A is simple and let K be the centre of A. Then R^* is commensurable with $Z(R)^* \times R^1$, where R^1 denotes the group of elements of reduced norm 1 of R. Moreover $\mathbb{R} \otimes_{\mathbb{Q}} K \cong \mathbb{R}^r \oplus \mathbb{C}^s$, where r is the number of embeddings of K in $\mathbb{R}$ and s is the number of pairs of non real embeddings of K in $\mathbb{C}$. These embeddings in $\mathbb{R}$ and pairs of embeddings in $\mathbb{C}$ correspond to infinite places of K (i.e. equivalence classes of archimedean valuations of K). If d is the degree of A then $\mathbb{R} \otimes_{\mathbb{Q}} A \cong M_d(\mathbb{R})^{r_1} \oplus M_{d/2}(\mathbb{H}(\mathbb{R}))^{r_2} \oplus M_d(\mathbb{C})^s$, where r_2 is the number of infinite places at which A is ramified and $r = r_1 + r_2$. Every embedding σ of K in $\mathbb{C}$ induces an embedding $\bar{\sigma} : A \to M_d(\mathbb{C})$ that maps R^1 into $\mathrm{SL}_d(\mathbb{C})$.

A totally definite quaternion algebra is a quaternion algebra A over a totally real number field K which is ramified at every infinite place, that is, $\sigma(K) \otimes_K A \cong \mathbb{H}(\mathbb{R})$ for every embedding $\sigma : K \to \mathbb{R}$; or equivalently $A = \left(\frac{a,b}{K}\right)$ with $\sigma(a), \sigma(b) < 0$ for every field homomorphism $\sigma : K \to \mathbb{R}$.

The simple algebra A is said to be of *Kleinian type* if there is an embedding $\psi : A \to M_2(\mathbb{C})$ such that $\psi(R^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ for some (any) order R in A, or equivalently if A is either a number field or A is a quaternion algebra and $\bar{\sigma}(R^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ for some embedding of K in $\mathbb{C}$. More generally, an *algebra of Kleinian type* [26] is by definition a direct sum of simple algebras of Kleinian type. So, a *finite group G* is of *Kleinian type* if and only if the rational group algebra $\mathbb{Q}G$ is of Kleinian type.

2 Equivalence of (A) and (B)

The equivalence between (A) and (B) is a direct consequence of the following more general theorem.

Theorem 2.1 *Let $A = \prod_{i=1}^n A_i$ be a finite dimensional rational algebra such that A_i is simple for every i. Let R be an order in A and for every i let R_i be an order in A_i. Then R^* is virtually a direct product of free-by-free groups if and only if R_i^1 is virtually free-by-free for every i.*

A group G is said to be *virtually indecomposable* if every subgroup of finite index of G is indecomposable as a direct product of two infinite groups. (Note that the terminology should not be confused with "having an indecomposable subgroup of finite index".)

To prove Theorem 2.1 we need the following lemma.

Lemma 2.2 *If C is a free-by-free group which is virtually indecomposable and not virtually abelian then $Z(C) = 1$.*

Proof. Suppose C is a free-by-free group. Then we may write $C = N \rtimes F$, with N and F free groups. We first prove that $Z(C) \subseteq N$. Suppose the contrary. Then it follows that $Z(F) \neq 1$ and thus F is cyclic. Therefore $\langle Z(C), N\rangle$ has finite index in C. As C is not virtually abelian, N is non-abelian, hence $\langle Z(C), N\rangle = Z(C) \times N$, contradicting the virtual indecomposability of C. So, indeed, $Z(C) \subseteq N$. If $Z(C) \neq 1$ then N is cyclic and $Z(C) \times F$ has finite index in C, again a contradiction. ∎

Proof of Theorem 2.1. Since R and $\prod_{i=1}^n R_i$ are two orders in A and R_i^* is commensurable with $Z(R_i)^* \times R_i^1$ for each i, one has that R^* and $\prod_{i=1}^n Z(R_i)^* \times R_i^1$ are commensurable. The sufficiency of the conditions is now clear.

Conversely, assume that the direct product $T = \prod_{x\in X} T_x$ is a subgroup of finite index of R^*, where every T_x is a non trivial free-by-free group. Since the virtual cohomological dimension of R^* is finite, X is finite and we can assume without loss of generality that every T_x is virtually indecomposable and either T_x is cyclic or is not virtually abelian. For every $x \in X$ let $\pi_x : T \to T_x$ denote the projection and let $Y = \{y \in X \mid T_y \text{ is not abelian}\}$. For every i let $S_i = R_i^1 \cap T$ and $Z_i = Z(R_i)^* \cap T$. Then S_i is a torsion free subgroup of finite index in R_i^1, Z_i is a torsion-free subgroup of finite index in $Z(R_i)^*$, $S_i \cap Z(R_i)^* = 1$ and $S = \prod_i Z_i \times S_i$ is a subgroup of finite index in T, because $R_i^1 \cap Z(R_i)$ is finite, $\langle Z(R_i)^*, R_i^1\rangle$ has finite index in R_i^* and T is a torsion-free subgroup of finite index in R^*.

We claim that if $\pi_z(S_j)$ is not abelian (and hence infinite) and $H = \left(\prod_{i\neq j} S_i\right) \times \left(\prod_i Z_i\right)$ then $\pi_z(H) = 1$. Indeed, $C = \pi_z(S)$ is a subgroup of finite index in T_z and therefore C satisfies the hypothesis of Lemma 2.2. Thus $\pi_z(S_j) \cap \pi_z(H) \subseteq Z(C) = 1$, because $(S_j, H) = 1$. Then $C = \pi_z(S_j) \times \pi_z(H)$ and from the virtual indecomposability of C one deduces that $\pi_z(H) = 1$. This finishes the proof of the claim.

We have to show that each R_i^1 is virtually free-by-free or equivalently that so is S_i. By [20, Theorem 1], S_i is virtually indecomposable. So either S_i is virtually cyclic, and we are done, or S_i is non-abelian. Assume that S_i is non-abelian. Hence there is $y \in Y$ such that $\pi_y(S_i)$ is non-abelian. Assume that $x \in X_i = \{x \in X \mid \pi_x(S_i) \neq 1\}$. Then $\pi_x(S_i)$ has finite index in T_x, for otherwise T_x is not cyclic and so there is at least one j such that $\pi_x(S_j)$ is non-abelian that gives, by the claim, $\pi_x(S_i) = 1$, a contradiction. Therefore S_i is a subgroup of finite index in $\prod_{x\in X_i} T_x$. As S_i is virtually indecomposable, $|X_i| = 1$ and therefore S_i is virtually free-by-free as wanted. ∎

3 (B) implies (C), (C) implies (D), and (E) implies (B)

It is well known that the virtual cohomological dimension of a free-by-free group is at most 2 and so (B) implies (C) is obvious.

(E) implies (B) is a direct consequence of the following lemma in which we collect known or recently established facts on the structure of the group of reduced norm one elements of an order in some simple rational algebras.

Lemma 3.1 *Let A be a simple finite dimensional rational algebra and R an order in A.*

1. *R^1 is finite if and only if A is a field or a totally definite quaternion algebra.*
2. *R^1 is virtually free non-abelian if and only if $A = M_2(\mathbb{Q})$.*
3. *If $A = M_2(\mathbb{Q}(\sqrt{-d}))$ with $d = 1, 2, 3, 7$ or 11 then R^1 is commensurable with a free-by-free group.*

Proof. See e.g. [30, Lemma 21.3] for 1 and [19] for 2.

3. Let O_d be the ring of integers of $\mathbb{Q}(\sqrt{-d})$. Then R^1 is commensurable with $\mathrm{SL}_2(O_d)$. So it is enough to show that $\mathrm{SL}_2(O_d)$ is virtually free-by-free. This is well known for $d = 3$, because $\mathrm{PSL}_3(O_d)$ has a subgroup of index 12 isomorphic to the figure eight knot group, a free-by-infinite cyclic group (see for example [23, page 137]). That $\mathrm{SL}_2(O_d)$ is virtually free-by-free for $d = 1, 2, 7$ or 11 has been proved in Lemmas 4.2 and 4.3 of [33]. ∎

In order to prove (C) implies (D) we classify in Proposition 3.2 the simple algebras of Kleinian type (correcting Proposition 3.1 in [26] where one possibility was missed by an error in the proof) and in Proposition 3.3 we classify the simple algebras A for which R^1 has virtual cohomological dimension at most 2 for an order R in A. Then (C) implies (D) follows at once from these two propositions.

Proposition 3.2 *A finite dimensional rational simple algebra A is of Kleinian type if and only if it is either a number field or a quaternion algebra which is not ramified at at most one infinite place.*

In particular, if A is non-commutative and of Kleinian type then the centre K of A has at most one pair of complex non-real embeddings and hence the order of every primitive root of unity in K is a divisor of 4 or 6.

Proof. Let R be an order in A and K the centre of A. Assume first that A is either a field or a quaternion algebra which is not ramified at at most one infinite place. If A is a field or a totally definite quaternion algebra then R^1 is finite by Lemma 3.1 and so A is of Kleinian type. If K is totally real then A is of Kleinian type by a theorem of Borel and Harish-Chandra [3] (see [22]). Otherwise K has exactly one pair of complex embeddings and A is ramified at all the real embeddings of K. Thus A is of Kleinian type by [7, Theorem 10.1.2].

Conversely, assume that A is of Kleinian type. Then A is either a number field or a quaternion algebra. In the remainder of the proof we assume that A is a quaternion algebra.

Let $\sigma_1, \ldots, \sigma_n$ be the set of representatives up to conjugation of the embeddings of K in $\mathbb{C}$. Each σ_i gives rise to an embedding $\overline{\sigma_i} : A \to A_i$ where $A_i = M_2(\mathbb{C})$ if σ_i is not real, $A_i = M_2(\mathbb{R})$ if σ_i is real and not ramified and $A_i = \mathbb{H}(\mathbb{R})$ otherwise. We consider A_i embedded in $M_2(\mathbb{C})$ in the obvious way. Then $\sigma_i(R^1) \subseteq SL_2(\mathbb{C})$. Let R be an order in A. Then $\overline{\sigma_i}(R^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ for some i, because by assumption A is of Kleinian type. We may assume that

$i = 1$. Assume that σ_l is either a non real embedding or a non ramified real embedding and let $f : A \to \prod_{j \neq k} A_j$ be the map given by $f(x) = (\overline{\sigma_j}(x))_{j \neq k}$. Then, by the Strong Approximation Theorem (see [27, Theorem 7.12] or [32, Theorem 4.3]), $f(R^1)$ is dense in $\prod_{j \neq k} A_j^1$ and therefore $k = 1$. This shows that A ramifies at at least $n-1$ places. Hence the result follows. ∎

Proposition 3.3 *Let A be a simple finite dimensional rational algebra and R an order in A. Let $\mathrm{vcd}(R^1)$ denote the virtual cohomological dimension of R^1. The following conditions hold.*

1. $\mathrm{vcd}(R^1) = 0$ *if and only if A is a field or a totally definite quaternion algebra.*

2. $\mathrm{vcd}(R^1) = 1$ *if and only if $A = M_2(\mathbb{Q})$.*

3. $\mathrm{vcd}(R^1) = 2$ *if and only if $A = M_2(K)$ with K an imaginary quadratic extension of the rationals or A is a quaternion algebra over a totally real number field which is not ramified at exactly one infinite place.*

Proof. Let $K = Z(A)$, r the number of embeddings of K in $\mathbb{R}$, s the number of of non-real embeddings of K in $\mathbb{C}$, r_1 the number of real embeddings of K at which A is ramified and $r_2 = r - r_1$ the number of real embeddings of K at which A is not ramified. Set $A = M_n(D)$ where D is a division ring of degree d. Notice that if d is odd then $r_1 = 0$.

The sufficiency of the respective conditions easily can be checked using the following formula for the virtual cohomological dimension of R^1 that can be deduced from the formulae on pages 220 and 222 and in Theorem 4 of [19]:

$$\begin{aligned} \mathrm{vcd}(R^1) &= r_2 \tfrac{(nd+2)(nd-1)}{2} + r_1 \tfrac{(nd-2)(nd+1)}{2} + s(n^2d^2-1) - n + 1 \\ &= r_2 nd + r\tfrac{(nd-2)(nd+1)}{2} + s(n^2d^2-1) - n + 1. \end{aligned} \tag{1}$$

Conversely, assume that A is not a field and $\mathrm{vcd}(R^1) \leq 2$. By (1) one has

$$r_2 nd + r\frac{(nd-2)(nd+1)}{2} + s(n^2d^2-1) \leq n+1. \tag{2}$$

Since A is not a field, $nd \geq 2$ and therefore the three summands on the left hand side of (2) are non-negative, which implies that each summand at most $n+1$. Hence, since $nd+1 \geq n+1$, we get that $s(nd-1)(nd+1) = s(n^2d^2-1) \leq n+1$ and thus it follows that either $s = 0$ or $d = s = 1$ and $n = 2$. In the latter case $r_1 = 0$ and since $s(n^2d^2-1) = n+1$, one has that $r_2 nd = 0$ so that $r_2 = 0$. Thus $A = M_2(K)$ where K is an imaginary quadratic extension of $\mathbb{Q}$.

Assume now that $s = 0$, that is, K is totally real. Now we use $r_2 nd \leq n+1$ to deduce that either (a) $r_2 = 0$, (b) $r_2 = d = 1$ or (c) $n = r_2 = 1$ and $d = 2$. We deal with each case separately.

(a) If $r_2 = 0$ then $r = r_1 \neq 0$, that is A is ramified at every infinite place of K. This implies that d is even. Furthermore

$$\frac{(nd-2)(nd+1)}{2} \leq r\frac{(nd-2)(nd+1)}{2} \leq n+1$$

and therefore $(nd-2)(nd+1) \leq 2n+2$. Thus $nd(nd-1) \leq 2n+4$ and so $n(d(nd-1)-2) \leq 4$. If $n \geq 2$ then $n(d(nd-1)-2) \geq 2(2\cdot 3-2) = 8$. Thus $n = 1$, that is $A = D$ is a division ring. Further $d(d-1) \leq 6$ and thus $d = 2$, because d is even. We conclude that A is a totally definite quaternion algebra.

(b) Assume that $r_2 = d = 1$. Then $r_1 = 0$, that is, $K = \mathbb{Q}$ and $r_2 nd = n$, so that $\frac{(n-2)(n+1)}{2} \leq 1$ and one deduces that $n = 2$. Thus $A = M_2(\mathbb{Q})$.

(c) Finally if $n = r_2 = 1$ and $d = 2$ then A is a quaternion algebra over a totally real number field which is not ramified at exactly one infinite place. ∎

The following corollary is an immediate consequence of Propositions 3.2 and 3.3. Of course it yields at once that (C) implies (D).

Corollary 3.4 *Let A be a finite dimensional simple rational algebra and R an order in A. If the virtual cohomological dimension of R^1 is at most 2 then A is of Kleinian type.*

Remark 3.5 By Proposition 3.2 there are six types of simple algebras of Kleinian type: (1) number fields; (2) totally definite quaternion algebras; (3) $M_2(\mathbb{Q})$; (4) $M_2(K)$, where K is an imaginary quadratic extension of the rationals; (5) quaternion division algebras over totally real number fields which are not ramified at exactly one infinite place; and (6) quaternion division algebras with exactly one pair of complex (non-real) embeddings which are ramified at all the real places.

Proposition 3.3 shows that the first five types correspond to the simple finite dimensional rational algebras A such that $\mathrm{vcd}(R^1) \leq 2$ for some (any) order R in A. In the sixth case $\mathrm{vcd}(R^1) = 3$ (by (3.1)).

"(D) implies (E)" of Theorem 1 (which will be proved in sections 4, 5 and 6) shows that if A is a simple component of $\mathbb{Q}G$ for G a finite group of Kleinian type then A is of one of the first four types of simple Kleinian algebras.

4 (F) implies (E)

To prove that (F) implies (E) we need to compute the simple components of $\mathbb{Q}G$, for G a finite group satisfying (F). This we will do using a method introduced in [24].

Let G be a finite group. For a subgroup H of G we set $\widehat{H} = \frac{1}{|H|}\sum_{h \in H} h$, an idempotent element in $\mathbb{Q}G$. If $g \in G$ then put $\widehat{g} = \widehat{\langle g \rangle}$. A *strong Shoda pair* of G is a pair (K, H) of subgroups of G such that $H \trianglelefteq K \trianglelefteq G$, K/H is cyclic and K/H is maximal abelian in $N_G(H)/H$. (The definition in [24] is more general but for our purposes we do not need such a generality.) If $K = H$ (and hence $K = G$), then let $\varepsilon(K, K) = \widehat{K}$; otherwise, let $\varepsilon(K, H) = \prod_{L \in M}(\widehat{H} - \widehat{L})$, where M is the set of minimal elements in the set of subgroups of K containing H properly. Finally, let $e(G, K, H)$ denote the sum of the different G-conjugates of $\varepsilon(K, H)$.

Let R be a ring and let G be a group. If $\rho \in \mathrm{Aut}(R)$ and $r \in R$ then we denote $\rho(r)$ as r^ρ. Recall from [25] that a *crossed product* of G over R with action $\sigma : G \to \mathrm{Aut}(R)$ and twisting $\tau : G \times G \to R^*$ is an associative ring $R * G = R *_\tau^\sigma G$ which contains R as a subring and a set of units $\{u_g \mid g \in G\}$ of $R * G$ such that $R * G = \oplus_{g \in G} u_g R$ (a free right R-module) and the product in $R * G$ is given by:

$$(u_g r)(u_h s) = u_{gh}\tau(g, h) r^{\sigma(h)} s, \quad (g, h \in G, r, s \in R).$$

Proposition 4.1 *[24] Let G be a finite group.*

1. *Assume that (K, H) is a strong Shoda pair of G. Let $N = N_G(H)$, $k = [K : H]$ and $n = [G : N]$. The following properties hold.*

    (a) $e = e(G, K, H)$ is a primitive central idempotent of $\mathbb{Q}G$.

(b) $\mathbb{Q}Ge$ *is isomorphic with* $M_n(\mathbb{Q}(\xi_k)*_\tau^\sigma N/K)$, *an* $n\times n$*-matrix ring over a crossed product of* N/K *over the cyclotomic field* $\mathbb{Q}(\xi_k)$, *with defining action and twisting given as follows: Let* x *be a generator of* K/H *and let* $\gamma : N/K \to N/H$ *be a left inverse of the natural epimorphism* $N/H \to N/K$. *Then*

$$\begin{array}{rl} \xi_k^{\sigma(a)} = \xi_k^i, & \text{if } x^{\gamma(a)} = x^i; \\ \tau(a,b) = \xi_k^j, & \text{if } \gamma(ab)^{-1}\gamma(a)\gamma(b) = x^j, \end{array}$$

for $a, b \in N/K$ *and integers* i *and* j.

(c) *The simple algebra* $\mathbb{Q}Ge$ *has degree* $[G : K]$.

(d) *The kernel of the natural group homomorphism* $G \to Ge$ *is* $\mathrm{Core}_G(H) = \bigcap_{g\in G} H^g$.

2. *If* G *is metabelian then every primitive central idempotent of* $\mathbb{Q}G$ *is of the form* $e(G, K, H)$ *for some strong Shoda pair* (K, H) *of* G.

Proof. Let θ be a linear character of K with kernel H. Then the induced character $\chi = \theta^G$ is irreducible and $e = e(G, K, H)$ is the unique primitive central idempotent of $\mathbb{Q}G$ such that $\chi(e) \neq 0$ [24]. This proves 1(a). The proofs of 1(b) and 2 can be found in [24] and 1(c) is a direct consequence of 1(b).

To prove 1(d) note that the kernel of $g \mapsto ge$ coincides with the kernel of χ. Since $H \trianglelefteq K \trianglelefteq G$, this kernel is $\{k \in K : \theta(gkg^{-1}) = 1, \text{ for all } g \in G\} = \bigcap_{g\in G}\{k \in K : \theta(gkg^{-1}) = 1, \text{ for all } g \in G\} = \bigcap_{g\in G} H^g = \mathrm{Core}_G(H)$. ∎

The following isomorphisms can be found in [6, p. 161-163], [30, Lemma 20.4] and [15]. (This can also can be verified using Proposition 4.1.)

$$\begin{array}{rcl} \mathbb{Q}C_n & \cong & \oplus_{d|n}\mathbb{Q}(\xi_d) \\ \mathbb{Q}D_{2n} & \cong & \mathbb{Q}(D_{2n}/D'_{2n}) \oplus \oplus_{d|n,2<d} M_2(\mathbb{Q}(\xi_d + \xi_d^{-1})) \\ \mathbb{Q}Q_{2^n} & \cong & \mathbb{Q}D_{2^{n-1}} \oplus \mathbb{H}(\mathbb{Q}(\xi_{2^{n-1}} + \xi_{2^{n-1}}^{-1})) \\ \mathbb{Q}D_{16}^- & \cong & 4\mathbb{Q} \oplus M_2(\mathbb{Q}) \oplus M_2(\mathbb{Q}(\sqrt{-2})) \\ \mathbb{Q}D_{16}^+ & \cong & 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus M_2(\mathbb{Q}(i)) \\ \mathbb{Q}\mathcal{D} & \cong & 8\mathbb{Q} \oplus M_2(\mathbb{Q}(i)) \\ \mathbb{Q}\mathcal{D}^+ & \cong & 4\mathbb{Q} \oplus 2\mathbb{Q}(i) \oplus 2M_2(\mathbb{Q}) \oplus 2M_2(\mathbb{Q}(i)) \end{array} \tag{3}$$

Next we prove three reduction lemmas.

Lemma 4.2 1. *The class of algebras of Kleinian type is closed under epimorphic images and semi-simple subalgebras.*

2. *The class of finite groups of Kleinian type is closed under subgroups and epimorphic images.*

3. *The class of finite groups satisfying condition (E) of Theorem 1 is closed under subgroups and epimorphic images.*

Proof. 1. Obviously the class of algebras of Kleinian type is closed under epimorphic images.

Let A be an algebra of Kleinian type and B a semi-simple subalgebra of A. If B_1 is a simple quotient of B then B_1 is isomorphic to a subalgebra of a simple quotient of A. In order to prove that B is an algebra of Kleinian type we thus may assume that A and B are simple and B is not a field. Since A is of Kleinian type and B is non-abelian, it is clear that A and B are quaternion algebras and that there is an order R in A and an embedding $\sigma : A \to M_2(\mathbb{C})$ such that $\sigma(R^1)$ is

a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then $S = R \cap B$ is an order in B and $\sigma(S^1) \subseteq \sigma(R^1)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$. This finishes the proof of 1.

2. This is a direct consequence of 1.

3. This can be proved imitating the proof of 1 and 2 and noticing that if A and B are as above (B non-commutative and simple) then $Z(B) \subseteq Z(A)$. ∎

For a finite group G we denote by $\mathcal{C}(G)$ the set of isomorphism classes of noncommutative simple quotients of $\mathbb{Q}G$. For simplicity we often represent $\mathcal{C}(G)$ by listing a set of representatives of its elements. For example, $\mathcal{C}(D_{16}^+) = \{M_2(\mathbb{Q}(i))\}$ and $\mathcal{C}(D_{16}^-) = \{M_2(\mathbb{Q}), M_2(\mathbb{Q}(\sqrt{-2}))\}$ (see (3)).

Lemma 4.3 *Let G be a finite group and A an abelian subgroup of G such that every subgroup of A is normal in G. Let $\mathcal{H} = \{H \mid H \text{ is a subgroup of } A \text{ with } A/H \text{ cyclic and } G' \not\subseteq H\}$. Then $\mathcal{C}(G) = \cup_{H\in\mathcal{H}}\mathcal{C}(G/H)$.*

Proof. Let H be a subgroup of A. By assumption, $H \trianglelefteq G$ and thus $\mathbb{Q}G = \mathbb{Q}G\widehat{H} \oplus \mathbb{Q}G(1-\widehat{H}) \cong \mathbb{Q}(G/H) \oplus \mathbb{Q}G(1-\widehat{H})$. It follows that $\mathcal{C}(G) \supseteq \cup_{H\in\mathcal{H}}\mathcal{C}(G/H)$. It is well known (and can be proved using Proposition 4.1) that the primitive central idempotents of $\mathbb{Q}A$ are the elements of the form $\varepsilon(A,H)$, where H runs through the set $\overline{\mathcal{H}} = \{H \le A \mid A/H \text{ is cyclic}\}$. Notice that each $\varepsilon(A,H)$ is central in $\mathbb{Q}G$ because H and A are normal in G. Thus $\{\varepsilon(A,H) \mid H \in \overline{\mathcal{H}}\}$ is a complete set of orthogonal central idempotents of $\mathbb{Q}G$ which are primitive central in $\mathbb{Q}A$ but not necessarily in $\mathbb{Q}G$. If f is a primitive central idempotent of $\mathbb{Q}G$ then there is $H \in \overline{\mathcal{H}}$ such that $f\varepsilon(A,H) = f$ and $f\varepsilon(A,K) = 0$ for each $H \ne K \in \overline{\mathcal{H}}$. Then $f \in \mathbb{Q}G\varepsilon(A,H)$ and hence $f \in \mathbb{Q}G\widehat{H} \cong \mathbb{Q}(G/H)$. Therefore $\mathbb{Q}Gf$ is a simple epimorphic image of $\mathbb{Q}(G/H)$. If $\mathbb{Q}Gf$ is non-commutative then G/H is non-abelian and thus $H \in \mathcal{H}$ and $\mathbb{Q}Gf \in \mathcal{C}(G/H)$. ∎

Lemma 4.4 *Let A be a finite abelian group of exponent d and G an arbitrary group.*

1. *If $d|2$ then $\mathcal{C}(A \times G) = \mathcal{C}(G)$.*
2. *If $d|4$ and $\mathcal{C}(G) \subseteq \left\{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(i))\right\}$ then $\mathcal{C}(A \times G) \subseteq \mathcal{C}(G) \cup \{M_2(\mathbb{Q}(i))\}$.*
3. *If $d|6$ and $\mathcal{C}(G) \subseteq \left\{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\xi_3))\right\}$ then $\mathcal{C}(A\times G) \subseteq \mathcal{C}(G)\cup\{M_2(\mathbb{Q}(\xi_3))\}$.*

Proof. Recall that if G_1 and G_2 are two groups then $\mathbb{Q}(G_1 \times G_2) \cong \mathbb{Q}G_1 \otimes_{\mathbb{Q}} \mathbb{Q}G_2$. Because of the first isomorphism in (3), this implies, in particular, that the simple quotients of $\mathbb{Q}A$ are of the form $\mathbb{Q}(\xi_k)$, for k a divisor of d. It then also follows that the elements of $\mathcal{C}(A \times H)$ are represented by the simple quotients of the algebras of the form $\mathbb{Q}(\xi_k) \otimes_{\mathbb{Q}} B$ for $k|d$ and $B \in \mathcal{C}(H)$. Hence, if $d = 2$ each $\mathbb{Q}(\xi_k) = \mathbb{Q}$ and thus we obtain that $\mathcal{C}(A \times G) = \mathcal{C}(G)$. If $d|4$ then $\mathbb{Q}A$ is isomorphic to a direct product of copies of $\mathbb{Q}$ and $\mathbb{Q}(i)$ and thus if, moreover, $\mathcal{C}(G) \subseteq \left\{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(i))\right\}$ then $\mathcal{C}(A \times G)$ is formed by elements of $\mathcal{C}(G)$ and simple quotients of $\mathbb{Q}(i) \otimes_{\mathbb{Q}} M_2(\mathbb{Q}) \cong M_2(\mathbb{Q}(i))$, $\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{H}(\mathbb{Q}) \cong M_2(\mathbb{Q}(i))$, $\mathbb{Q}(i) \otimes_{\mathbb{Q}} \left(\frac{-1,-3}{\mathbb{Q}}\right) \cong M_2(\mathbb{Q}(i))$ and $\mathbb{Q}(i) \otimes_{\mathbb{Q}} M_2(\mathbb{Q}(i)) \cong 2M_2(\mathbb{Q}(i))$. This proves 1 and 2. To prove 3 one argues similarly using that if $d|6$ then every simple quotient of $\mathbb{Q}A$ is isomorphic to either $\mathbb{Q}$ or $\mathbb{Q}(\xi_3)$ and $\mathbb{Q}(\xi_3) \otimes_{\mathbb{Q}} M_2(\mathbb{Q}) \cong \mathbb{Q}(\xi_3) \otimes_{\mathbb{Q}} \mathbb{H}(\mathbb{Q}) \cong \mathbb{Q}(\xi_3) \otimes_{\mathbb{Q}} \left(\frac{-1,-3}{\mathbb{Q}}\right) \cong M_2(\mathbb{Q}(\xi_3))$. ∎

Now we compute $\mathcal{C}(G)$ for some of the groups G listed in (F) of Theorem 1.

Lemma 4.5 *1.* $\mathcal{C}(\mathcal{W}_{1n}) = \{M_2(\mathbb{Q})\}$.

2. $\mathcal{C}(\mathcal{W}) = \mathcal{C}(\mathcal{W}_{2n}) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q})\}$.

3. $\mathcal{C}(\mathcal{V}), \mathcal{C}(\mathcal{V}_{1n}), \mathcal{C}(\mathcal{V}_{2n}), \mathcal{C}(\mathcal{U}_1), \mathcal{C}(\mathcal{U}_2), \mathcal{C}(\mathcal{T}_{1n}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$.

4. $\mathcal{C}(\mathcal{T}), \mathcal{C}(\mathcal{T}_{2n}), \mathcal{C}(\mathcal{T}_{3n}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i)), \mathbb{H}(\mathbb{Q}(\sqrt{2})), M_2(\mathbb{Q}(\sqrt{-2}))\}$.

5. *Let $G = M \rtimes P$ be a semidirect product, where M is a non trivial elementary abelian 3-group. Suppose the centralizer $Q = \mathrm{C}_P(M)$ of M in P has index 2 in P and $m^p = m^{-1}$ for every $p \in P \setminus Q$.*

   (a) *If $P = \langle x \rangle$ is cyclic of order 2^n then $\mathcal{C}(G) = \mathcal{C}(G/\langle x^2 \rangle) \cup \left\{ \left(\frac{\xi_{2^{n-1}}, -3}{\mathbb{Q}(\xi_{2^{n-1}})} \right) \right\}$. In particular, if $P = C_8$ then $\mathcal{C}(G) = \left\{ M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}} \right), M_2(\mathbb{Q}(i)) \right\}$.*

   (b) *If $P = \mathcal{W}_{1n}$ and $Q = \langle y_1, \ldots, y_n, t_1, \ldots, t_n, x^2 \rangle$ then $\mathcal{C}(G) = \left\{ M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}} \right), M_2(\mathbb{Q}(\xi_3)) \right\}$.*

   (c) *If $P = \mathcal{W}_{21}$ and $Q = \langle y_1^2, x \rangle$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}(\sqrt{3})), M_2(\mathbb{Q}(i)), M_2(\mathbb{Q}(\xi_3))\}$.*

Proof. We use the notation and presentation of the groups as given in part (F) of Theorem 1. For the groups $\mathcal{W}_{11}$, $\mathcal{V}_{11}$ and $\mathcal{T}_{11}$ we put $y = y_1$ and $t = t_1$.

Throughout the proof we will use Lemma 4.3, Lemma 4.4 and (3). For several of the groups G mentioned in the statement of the lemma, we will identify a group A satisfying the conditions of Lemma 4.3. By $\mathcal{H}$ we then denote the set (depending on A) considered in Lemma 4.3. So that $\mathcal{C}(G) = \cup_{H \in \mathcal{H}} \mathcal{C}(G/H)$.

1 and 2. For $\mathcal{W}_{11}$, let $A = \langle x^2, t \rangle = Z(\mathcal{W}_{11})$. If $H \in \mathcal{H}$ then $H = \langle x^2 \rangle$ or $\langle tx^2 \rangle$ and hence $\mathcal{W}_{11}/H \cong D_8$. Thus $\mathcal{C}(\mathcal{W}_{11}) = \{M_2(\mathbb{Q})\}$.

For $\mathcal{W}$, let $A = Z(\mathcal{W}) = \langle x^2, y^2, t \rangle$. If $H \in \mathcal{H}$, then $\mathcal{W}/H$ is a non-abelian group of order 8 and thus it is isomorphic to D_8 or Q_8. Hence $\mathcal{C}(\mathcal{W}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q})\}$. The converse inclusion is clear, because D_8 and Q_8 are epimorphic images of $\mathcal{W}$.

Since $\mathcal{W}_{21}$ is an epimorphic image of $\mathcal{W}$, and D_8 and Q_8 are epimorphic images of $\mathcal{W}_{21}$, one has that $\mathcal{C}(\mathcal{W}_{21}) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q})\}$.

For $G = \mathcal{W}_{mn}$ with $m = 1$ or 2 and $n \geq 2$, consider $A = G' = \langle t_1, \ldots, t_n \rangle$. Then, using the relations $y^2 = (y, x)^{m-1}$ for every $y \in \langle y_1, \ldots, y_n \rangle$, one deduces that $G/H \cong C_2^{n-1} \times \mathcal{W}_{m1}$ for every $H \in \mathcal{H}$ and thus $\mathcal{C}(\mathcal{W}_{mn}) = \mathcal{C}(\mathcal{W}_{m1})$.

3. For $\mathcal{V}$, take $A = Z(\mathcal{V}) = \langle x^2, y^2, (y, x) \rangle$ and let $H \in \mathcal{H}$. If $[A : H] = 2$ then $\mathcal{V}/H$ has order 8 and then $\mathcal{C}(\mathcal{V}/H)$ is either $\{M_2(\mathbb{Q})\}$ or $\{\mathbb{H}(\mathbb{Q})\}$. Otherwise A/H is cyclic of order 4 and therefore $x^4 \notin H$ or $y^4 \notin H$. Thus $\mathcal{V}/H$ is a group of order 16, exponent 8 and with commutator subgroup of order 2. This implies that $\mathcal{V}/H \cong D_{16}^+$ and $\mathcal{C}(\mathcal{V}/H) = \{M_2(\mathbb{Q}(i))\}$. Thus $\mathcal{C}(\mathcal{V}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$.

For $G = \mathcal{T}_{11}$, we need a different argument. Consider $K = \langle t, y, x^2 \rangle$, an abelian subgroup of index 2 in G, and the following eight subgroups of K:

$$\begin{array}{llll} H_1 = \langle y, x^2 \rangle, & H_2 = \langle y, x^2 t^2 \rangle, & H_3 = \langle y, tx^2 \rangle, & H_4 = \langle y, tx^{-2} \rangle, \\ H_5 = \langle yt^2, x^2 \rangle, & H_6 = \langle yt^2, x^2 t^2 \rangle, & H_7 = \langle yt^2, tx^2 \rangle, & H_8 = \langle yt^2, tx^{-2} \rangle \end{array}$$

A straightforward calculation shows that $K = N_G(H_i)$ and K/H_i is cyclic (generated by the class of t) of order 4 for every i, so that (K, H_i) is a strong Shoda pair of G for every i. By Proposition 4.1, each $e_i = e(G, K, H_i) = \varepsilon(K, H_i) + \varepsilon(K, H_i^x)$ is a primitive central idempotent of $\mathbb{Q}G(1 - \widehat{t^2})$ and $\mathbb{Q}Ge_i \cong M_2(\mathbb{Q}(i))$. Furthermore the 16 subgroups of the form H_i and H_i^x are pairwise different.

This implies that the 8 primitive central idempotents e_i are pairwise different and hence $\mathbb{Q}G = \mathbb{Q}G\widehat{t^2} \oplus \oplus_{i=1}^8 \mathbb{Q}e_i \cong \mathbb{Q}(G/\langle t^2\rangle) \oplus 8M_2(\mathbb{Q}(i))$. Since $G/\langle t^2\rangle$ is an epimorphic image of $\mathcal{V}$, one concludes that $\mathcal{C}(\mathcal{T}_{11}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$. Actually $\mathcal{C}(\mathcal{V}) = \mathcal{C}(\mathcal{T}_{11}) = \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$ because $\mathcal{W}$ is an epimorphic image of $\mathcal{T}_{11}$ and $\mathcal{V}$.

For $G = \mathcal{V}_{1n}, \mathcal{V}_{2n}, \mathcal{U}_1, \mathcal{U}_2$ or $\mathcal{T}_{1n}$, we consider $A = G'$ and let $H \in \mathcal{H}$. If $G = \mathcal{T}_{1n}$ then G/H is an epimorphic image of $C_4^{n-1} \times \mathcal{T}_{11}$, and otherwise G/H is an epimorphic image of $C_2^k \times \mathcal{V}$ for some k. We conclude that $\mathcal{C}(G) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$.

4. For $G = \mathcal{T}$, take $A = Z(G) = \langle t^2, ty^2\rangle$. Let $H \in \mathcal{H}$. If either $t^2 \in H$ or $y^4 \in H$ then G/H is an epimorphic image of either $\mathcal{V}$ or $\mathcal{T}_{11}$ and so $\mathcal{C}(G/H) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$. Otherwise, that is, if $t^2, y^4 \notin H$, then $t^2y^4 \in H$ and hence $H = \langle ty^2\rangle$ or $H = \langle t^{-1}y^2\rangle$. Then G/H is isomorphic to either Q_{16} or D_{16}^- and $\mathcal{C}(G/H) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}(\sqrt{2})), M_2(\mathbb{Q}(\sqrt{-2}))\}$. We conclude that $\mathcal{C}(\mathcal{T}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i)), \mathbb{H}(\mathbb{Q}(\sqrt{2})), M_2(\mathbb{Q}(\sqrt{-2}))\}$.

For $\mathcal{T}_{21}$, take $A = Z(\mathcal{T}_{21}) = \langle t^2, x^2\rangle$ and $H \in \mathcal{H}$. If $t^2 \in H$ or $t^2x^2 \in H$ then G/H is an epimorphic image of $\mathcal{V}$ or $\mathcal{T}$. Otherwise, $H = \langle x^2\rangle$ and hence $G/H = D_{16}^-$. So $\mathcal{C}(\mathcal{T}_{21}) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i)), \mathbb{H}(\mathbb{Q}(\sqrt{2})), M_2(\mathbb{Q}(\sqrt{-2}))\}$.

For $G = \mathcal{T}_{2n}$, taking $A = G'$ and having in mind that $(y, x)y^2 = 1$ for every $y \in \langle y_1, \dots, y_n\rangle$, one deduces that G/H is an epimorphic image of $T_{21} \times C_2^{n-1}$ for every $H \in \mathcal{H}$. Hence, by the previous paragraph, $\mathcal{C}(\mathcal{T}_{2n}) = \mathcal{C}(\mathcal{T}_{21})$.

Finally, for $G = \mathcal{T}_{3n}$, take $A = G'$ and let $H \in \mathcal{H}$. If $t_1^2 \in H$ then G/H is an epimorphic image of $\mathcal{T}_{3n}/\langle t_1^2\rangle$ which in turn is an isomorphic image of $\mathcal{V}_{1n}$ and thus $\mathcal{C}(G/H) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$. Otherwise, the image $\overline{t_i}$ in G/H of each t_i belongs to $\langle \overline{t_1}\rangle$. Furthermore, $\overline{t_1}$ has order 4 and $\overline{t_i}$ has order at most 2, for $i \geq 2$. Thus $\overline{t_i} \in \langle t_1^2\rangle$. For $i \geq 2$, let z_i be the natural image of y_i in G/H if $t_i = 1$ and the natural image of $y_1^2 y_i$, otherwise. Then $Z = \langle z_2, \dots, z_n\rangle$ is central in G/H of exponent at most 2 and G/H is an epimorphic image of $\mathcal{T} \times Z$. Thus $\mathcal{C}(G/H) \subseteq \mathcal{C}(\mathcal{T})$.

5. Let $G = M \rtimes P$ be as in statement 5. Applying Lemma 4.3, with $A = M$, we may assume, without loss of generality, that M is cyclic of order 3, generated by m, say.

(a) Assume $P = \langle x\rangle$ is cyclic of order 2^n. Then $(K = \langle m, x^2\rangle, 1)$ is a strong Shoda pair of G and so $e = e(G, K, 1)$ is a primitive central idempotent of $\mathbb{Q}G$. Applying Proposition 4.1 one has $\mathbb{Q}Ge = \mathbb{Q}(\xi)[u | u^2 = \xi^3, u^{-1}\xi u = \xi^s]$, where $\xi = \xi_{3\cdot 2^{n-1}}$ and s is an integer such that $s \equiv -1 \mod 3$ and $s \equiv 1 \mod 2^{n-1}$. Let $\omega = \xi^{2^{n-1}}$, a third root of unity. Then $j^2 = -3$ and $ju = -uj$, where $j = 1 + 2\omega$. This shows that $\mathbb{Q}Ge \cong \left(\frac{\xi_{2^{n-1}}, -3}{\mathbb{Q}(\xi_{2^{n-1}})}\right)$. Since $e + (1 - \widehat{G'})\widehat{x^2} = 1 - \widehat{G'}$, one concludes that $\mathcal{C}(G) = \mathcal{C}(G/\langle x^2\rangle) \cup \{\mathbb{Q}Ge\}$.

In particular, if $n = 1$ then $\mathcal{C}(G) = \{\left(\frac{1,-3}{\mathbb{Q}}\right) = M_2(\mathbb{Q})\}$, if $n = 2$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right)\}$ and if $n = 3$ then $\mathcal{C}(G) = \{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), \left(\frac{i,-3}{\mathbb{Q}(i)}\right) = M_2(\mathbb{Q}(i))\}$. The equality $M_2(\mathbb{Q}(i)) = \left(\frac{i,-3}{\mathbb{Q}(i)}\right)$ holds because the equation $iX^2 - 3Y^2 = 1$ has the solution $X = 1 + i$ and $Y = i$.

(b) Assume $P = \mathcal{W}_{1n}$. Applying Lemma 4.3 with $A = P'$, we may assume that $n = 1$, because $G/H \cong C_2^{n-1} \times (M \rtimes \mathcal{W}_{11})$ for every $H \in \mathcal{H}$. So suppose $P = \mathcal{W}_{11}$ and $Q = \langle x^2, y = y_1, t = t_1\rangle$. Let S be a simple quotient of $\mathbb{Q}G$. Put $T = G/\langle t\rangle$ and $e = (1 - \widehat{m})(1 - \widehat{t})$, a central idempotent of $\mathbb{Q}G$. Notice that $G/\langle m\rangle \cong P$ and $T \cong C_2 \times (C_3 \rtimes C_4)$. If S is a quotient of $\mathbb{Q}G(1 - e)$ then S is a quotient of either $\mathbb{Q}G\widehat{m} \cong \mathbb{Q}(G/\langle m\rangle) \cong \mathbb{Q}P$ or $\mathbb{Q}G\widehat{t} \cong \mathbb{Q}T \cong \mathbb{Q}(C_2 \times (C_3 \rtimes C_4))$. Then $S \cong M_2(\mathbb{Q})$ or $\left(\frac{-1,-3}{\mathbb{Q}}\right)$, by the previous paragraph and statement 1. Otherwise, S is a quotient of $\mathbb{Q}Ge$. Notice that $e = e(G, K, H_1) + e(G, K, H_2)$, where $K = \langle Q, m\rangle$, $H_1 = \langle x^2, y\rangle$ and $H_2 = \langle tx^2, y\rangle$ and (K, H_1) and (K, H_2) are two strong Shoda pairs of G. Clearly $[K : H_1] = [K : H_2] = 6$ and $K = N_G(H_1) = N_G(H_2)$. Hence, because of Proposition 4.1, it follows that $\mathbb{Q}Ge \cong 2M_2(\mathbb{Q}(\sqrt{-3}))$, and therefore $S \cong M_2(\mathbb{Q}(\sqrt{-3}))$.

(c) Assume $P = \mathcal{W}_{12}$ and $Q = \langle x, y^2 \rangle$. Let S be a simple quotient of $\mathbb{Q}G$. Again, put $T = G/\langle t \rangle$ and $e = (1-\widehat{m})(1-\widehat{t})$. Then $G/\langle m \rangle \cong P$ and $T \cong C_4 \times (C_3 \rtimes C_2)$. Thus, if S is a quotient of $\mathbb{Q}G(1-e)$ then S is a quotient of either $\mathbb{Q}G\widehat{m} \cong \mathbb{Q}(G/\langle m \rangle) \cong \mathbb{Q}P$ or $\mathbb{Q}G\widehat{t} \cong \mathbb{Q}T \cong \mathbb{Q}(C_4 \times (C_3 \rtimes C_2))$. Then, as in the proof of (b), S is isomorphic to either $M_2(\mathbb{Q})$, $\mathbb{H}(\mathbb{Q})$ or $M_2(\mathbb{Q}(i))$. Otherwise, S is a quotient of $\mathbb{Q}Ge$. In this case, $e = e(G, K, H_1) + e(G, K, H_2)$, where $K = \langle Q, m \rangle$, $H_1 = \langle x \rangle$, $H_2 = \langle x^2 y^2 \rangle$, and (K, H_1) and (K, H_2) are two strong Shoda pairs of G. Since $[K : H_1] = 6$ and $K = N_G(H_1)$ we get that $\mathbb{Q}Ge(G, K, H_1) \cong M_2(\mathbb{Q}(\sqrt{-3}))$, as desired. Because $N_G(H_2) = G$, we deduce from Proposition 4.1 that $\mathbb{Q}Ge(G, K, H_2)$ is the simple algebra given by the following presentation: $S = \mathbb{Q}(\xi)[u | u^2 = -1, u^{-1}\xi u = \xi^{-1}]$, with $\xi = \xi_{12}$. Let $i = \xi^3$, $j = u$ and $a = \xi + \xi^{-1} \in Z(S)$. Clearly $i^2 = u^2 = -1$, $a^2 = 3$ and $ji = -ij$. Therefore $S \cong \mathbb{H}(\mathbb{Q}(\sqrt{3}))$. Thus $S \cong M_2(\mathbb{Q}(\sqrt{-3}))$ or $\mathbb{H}(\mathbb{Q}(\sqrt{3}))$. ∎

We are ready to prove (F) implies (E). So, let G be a finite group satisfying (F). By Lemma 4.2, to prove that G satisfies (E) one may assume that $G = A \times H$ for A and H satisfying one of the conditions 1-4 in (F). We have to show that the elements of $\mathcal{C}(G)$ either are totally definite quaternion algebras or are of the form $M_2(\mathbb{Q}(\sqrt{-d}))$ for $d = 0, 1, 2$ or 3. Using Lemma 4.4 and Lemma 4.5, one obtains the following five statements, and hence the result follows. If either condition 1 or condition 2 holds then $\mathcal{C}(G) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i))\}$. If condition 3 holds then $\mathcal{C}(G) \subseteq \{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}), M_2(\mathbb{Q}(i)), \mathbb{H}(\mathbb{Q}(\sqrt{2})), M_2(\mathbb{Q}(\sqrt{-2}))\}$. If condition 4 holds then $\mathcal{C}(G)$ is contained in either $\left\{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(i))\right\}$, $\left\{M_2(\mathbb{Q}), \left(\frac{-1,-3}{\mathbb{Q}}\right), M_2(\mathbb{Q}(\sqrt{-3}))\right\}$ or $\{M_2(\mathbb{Q}), \mathbb{H}(\mathbb{Q}(\sqrt{3})), M_2(Q(i)), M_2(\mathbb{Q}(\sqrt{-3}))\}$, depending on the respective cases.

5 (D) implies (F), for nilpotent groups

In the remainder of the paper we prove (D) implies (F), or equivalently we classify the groups of Kleinian type as the epimorphic images of the groups listed in (F). In this section we do this for finite groups that are nilpotent.

We start with two lemmas which provide information on the groups of Kleinian type.

Lemma 5.1 *Let G be a finite non-abelian group of Kleinian type. The following properties hold.*

1. *Either $G/Z(G)$ is elementary abelian of order 8 or G has an abelian normal subgroup of index 2. In particular, G is metabelian and has a nilpotent subgroup of index at most 2.*
2. *Every primitive central idempotent of $\mathbb{Q}G$ is of the form $e = e(G, K, H)$ for some strong Shoda pair (K, H) of G. Moreover, for such a primitive central idempotent e one has*
   (a) *$[G : K] \leq 2$;*
   (b) *if H is not normal in G then $K = N_G(H)$ and $[K : H]$ divides 4 or 6, and*
   (c) *if $\mathbb{Q}Ge$ is not a division ring then $[K : H]$ divides 8 or 12 and $\mathbb{Q}Ge$ is isomorphic to $M_2(\mathbb{Q}(\sqrt{-d}))$ for $d = 0, 1, 2$ or 3.*
3. *If a dihedral group D_{2n} is an epimorphic image of a subgroup of G then n divides 4 or 6.*
4. *$G = G_3 \rtimes G_2$ where G_3 is an elementary abelian 3-group (possibly trivial), G_2 is a 2-group and the kernel of the action of G_2 on G_3 has index at most 2 in G_2.*
5. *The exponent of $Z(G)$ is a divisor of 4 or 6.*

6. $Z(G) \cap G' = \{t \in G' \mid t^2 = 1\}$. *Furthermore, if* $t \in G'$ *and* $x \in G$ *then either* $t^x = t$ *or* $t^x = t^{-1}$. *In particular, every subgroup of* G' *is normal in* G.

Proof. 1. Since every simple quotient of $\mathbb{Q}G$ has degree at most 2 (see the definition of groups and algebras of Kleinian type), the irreducible character degrees of G are 1 and 2. By [1] this implies that either $G/Z(G)$ is elementary abelian of order 8 or G has an abelian subgroup of index 2. In the first case G is central-by-abelian, and hence nilpotent and metabelian. In the second case, obviously G is also metabelian and it has a nilpotent (in fact abelian) subgroup of index 2.

2. That every primitive central idempotent of $\mathbb{Q}G$ is of the form $e = e(G, K, H)$ for some strong Shoda pair of G is a consequence of Proposition 4.1 and the fact that G is metabelian. Let $e = e(G, K, H)$ for (K, H) a strong Shoda pair of G.

The inequality $[G : K] \leq 2$ is a straightforward consequence of the fact that $[G : K]$ equals the degree of $\mathbb{Q}Ge$ (Proposition 4.1). Let $k = [K : H]$. Since $K \leq N_G(H)$, we get that either $K = N_G(H)$ or $G = N_G(H)$. Therefore, if H is not normal in G then $K = N_G(H)$ and $\mathbb{Q}Ge = M_2(\mathbb{Q}(\xi_k))$. By Proposition 3.2, $\varphi(k) = [\mathbb{Q}(\xi_k) : \mathbb{Q}] \leq 2$ and therefore k divides 4 or 6. This proves (b) and it also proves (c) if H is not normal in G. If $\mathbb{Q}Ge$ is not a division ring and H is normal in G then, by Proposition 4.1, $\mathbb{Q}Ge \cong M_2(F)$ where F is a subfield of index 2 in $\mathbb{Q}(\xi_k)$. Furthermore, Remark 3.5 implies that F is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. Hence $\varphi(k) = 2[F : \mathbb{Q}]$, a divisor of 4. If $\varphi(k) \neq 4$ then k divides 4 or 6 and therefore $A = M_2(\mathbb{Q})$. Otherwise $k = 5, 8, 10$ or 12. If $5|k$ then necessarily $F = \mathbb{Q}(\sqrt{5})$, a contradiction. Thus $k = 8$ or 12 and therefore $A = M_2(\mathbb{Q}(\sqrt{-d}))$ for $d = 1, 2$ or 3. This finishes the proof of 2.

3. By (3), $\mathbb{Q}D_{2n}$ has an epimorphic image isomorphic to $M_2(\mathbb{Q}(\xi_n + \xi_n^{-1}))$. This algebra is of Kleinian type, by Lemma 4.2. Therefore $\mathbb{Q}(\xi_n + \xi_n^{-1}) = \mathbb{Q}$, by Remark 3.5 and this implies that n divides 4 or 6.

4. Let E be the set of primitive central idempotents e of $\mathbb{Q}G$ such that $\mathbb{Q}Ge$ is non-commutative. Let $e \in E$ and $z \in Z(G)$. By Proposition 3.2 the order of ze divides 4 or 6 and thus the exponent of $Z(G)$ divides 12.

By 1, G has a nilpotent subgroup of index at most 2. Therefore, $G = G_{2'} \rtimes G_2$, where $G_{2'}$ is a nilpotent subgroup of odd order of G, G_2 is a Sylow 2-subgroup of G and the kernel of the action of G_2 on $G_{2'}$ has index at most 2 in G_2. We have to show that $G_{2'}$ is an elementary abelian 3-group. We argue by contradiction. So, let $a \in G_{2'}$ be of order q, where q is either 9 or $q > 3$ and prime. Since the exponent of $Z(G)$ divides 12, a is not central in G and so there is $x \in G_2$ such that $a^x \neq a$. Put $b = aa^x$. Assume that $b = 1$. Then $\langle a, x\rangle/\langle x^2\rangle \cong D_{2q}$ and thus D_{2q} is of Kleinian type by Lemma 4.2, contradicting 3. Therefore b is a non-trivial central element of odd order. Hence b has order 3 and a has order 9. If $b \in \langle a\rangle$, then $b = a^{\pm 3}$ and hence $a^x = a^2$ or $a^x = a^{-4}$. Then $a = a^{x^2} = a^4$ or $a = a^{x^2} = a^7$, a contradiction. Thus $\langle a, x\rangle/\langle x^2\rangle = (\langle a\rangle_9 \times \langle b\rangle_3) \rtimes \langle x\rangle_2$, with $a^x = a^{-1}b$ and $b^x = b$. Then $\langle a, x\rangle/\langle b, x^2\rangle \cong D_{18}$, again a contradiction.

5. Since the exponent of $Z(G)$ divides 12 it is enough to show that $Z(G)$ does not have elements of order 12. By means of contradiction assume that $a \in Z(G)$ has order 12. Let $\varepsilon = \varepsilon(\langle a\rangle, 1)$ (see the notation introduced in Section 4). If $\varepsilon(1 - \widehat{G'}) \neq 0$ then there is a (necessarily injective) non-zero homomorphism $\mathbb{Q}(\xi_{12}) \cong \mathbb{Q}\langle a\rangle\varepsilon \to Z(A)$ for some non-commutative simple quotient A of $\mathbb{Q}G$. This implies that A has a central root of unity of order 12, contradicting Proposition 4.1. Thus $\varepsilon = \varepsilon\widehat{G'}$. If $H = \langle a\rangle \cap G'$ then $0 \neq \varepsilon = \varepsilon\widehat{G'} = \varepsilon\widehat{H}\widehat{G'}$. If $H \neq 1$ then $0 \neq \varepsilon\widehat{H} = (1 - \widehat{a^4})(1 - \widehat{a^6}) = 0$ because H contains either a^4 or a^6. Thus $H = 1$ and so $\varepsilon = \varepsilon\widehat{G'} = \frac{1}{|G'|}\varepsilon\sum_{g \in G'} g$. Since all the elements of G with non-zero coefficient in ε belongs to $\langle a\rangle$, the last formula implies that $G' \subseteq \langle a\rangle$. Thus $G' = 1$, contradicting the fact that G is non-abelian.

6. If $G/Z(G)$ is elementary abelian then $G' \subseteq Z(G)$. It follows that, for $a, b \in G$ we get that

$b^2a = ab^2 = batb = b^2at^2$, where $t = (a,b)$. Then $t^2 = 1$ and the statement follows.

So assume that $G/Z(G)$ is not elementary abelian. From 1, we then have that G has an abelian subgroup N of index 2. Let $a \in G \setminus N$. Then $t^x = t$ if $x \in N$. If $x \in G \setminus N$ then $x = na$ for some $n \in N$ and therefore $t^x = t^a$. Moreover $a^2 \in Z(G)$ and then, for every $g \in G$, one has $1 = (g,a^2) = (g,a)(g,a)^{a^{-1}} = (g,a)(g,a)^a$. Thus $(g,a)^a = (g,a)^{-1}$. On the other hand if $n,m \in N$ then $(na,ma) = (na,m)(na,a)^{m^{-1}} = (a,m)^{n^{-1}}(n,m)\left((a,a)^{n^{-1}}(n,a)\right)^{m^{-1}} = (a,m)(n,a) = (m,a)^{-1}(n,a)$. Thus if $t \in G'$ then $t = (n_1,a)^{\alpha_1} \cdots (n_k,a)^{\alpha_k}$ for some $n_i \in N$ and $\alpha_i \in \mathbb{Z}$ and $t^x = (n_1,a)^{-\alpha_1} \cdots (n_k,a)^{-\alpha_k} = t^{-1}$. So we have shown that $t^x = t$ if $x \in N$ and $t^x = t^{-1}$ otherwise. Therefore $t \in Z(G)$ if and only if $t^2 = 1$. ∎

Lemma 5.2 *If G is a finite non-abelian 2-group of Kleinian type then the following properties hold.*

1. *The exponent of G is at most 8.*
2. *G' is abelian of exponent at most 4.*
3. *$\langle (G,G') \rangle = G'^2 \subseteq Z(G)$.*
4. *$G_x = \langle (x,g) \mid g \in G \rangle$ is a normal subgroup of G for all $x \in G$. Moreover, if $G' \neq G_x$ then $x^4 \in G_x$.*
5. *If $x,y \in G$ and $t = (y,x)$ then one of the following conditions holds:*
   - (a) *$(x,t) = 1$, $(y,x^2) = t^2$ and $(y,tx^2) = 1$ or*
   - (b) *$(x,t) \neq 1$ and $(y,x^2) = 1$.*

Proof. Let E be the set of primitive central idempotents e of $\mathbb{Q}G$ such that $\mathbb{Q}Ge$ is non-commutative. It is well known that $1 - \widehat{G'} = \sum_{e \in E} e$ (see for example [4]). Notice that the coefficient of 1 in $g(1-\widehat{G'})$ is 0, if $g \notin G'$, and it is $-\frac{1}{|G'|}$, if $1 \neq g \in G'$. However the coefficient of 1 in $1 - \widehat{G'}$ is $1 - \frac{1}{|G'|} > 0$ because, by assumption, G' is non trivial. This shows that the natural group homomorphism $G \to G(1-\widehat{G'})$, mapping $g \in G$ onto $g(1-\widehat{G'})$, is injective, and hence so is the natural group homomorphism $f : G \to \prod_{e \in E} Ge$.

1. We prove the statement by contradiction. So suppose G is a non-abelian 2-group of Kleinian type of minimal order such that the exponent of G is greater than 8. Let $g \in G$ be of order 16.

Then, there is $e \in E$ such that ge has order 16 and, by the minimality of G, G is isomorphic to Ge. By Proposition 4.1, there is a strong Shoda pair (K,H) of G such that $e = e(G,K,H)$, $[G:K] = 2$ and $\mathrm{Core}_G(H) = 1$. Then $A = \mathbb{Q}Ge$ has a subfield isomorphic to $\mathbb{Q}(\xi_{16})$. Since A is a quaternion algebra, the dimension of the centre of A is at least $\varphi(16)/2 = 4$. Hence, by statement 2 of Proposition 5.1, A is a division algebra. It follows from statement (b) of Theorem 4.1 implies that $H \trianglelefteq G$, that is, $H = \mathrm{Core}_G(H) = 1$. Thus K is a cyclic subgroup of index 2 in G. Hence, as mentioned in the preliminaries, G is isomorphic to either $D_{2^{k+1}}$, $D^+_{2^{k+1}}$, $D^-_{2^{k+1}}$, or $Q_{2^{k+1}}$. Since A is a non-commutative division ring containing $\mathbb{Q}(\xi_{16})$, $G = Q_{2^{k+1}}$, one has that $k \geq 4$ (see (3)). Thus D_{16} is a quotient of G, in contradiction with statement 3 of Lemma 5.1.

2. That G' is abelian is a consequence of statement 1 of Lemma 5.1. We prove by contradiction that G' has exponent at most 4. So, assume that G is a non-abelian 2-group of Kleinian type of minimal order with a commutator $t = (y,x)$ of order greater than 4. By the minimality of the order of G, one has that $G = \langle x,y \rangle$. By statement 6 of Lemma 5.1, $t \in G' \setminus Z(G)$. Hence $G/Z(G)$

is non-abelian and statement 1 of Lemma 5.1 implies that G has an abelian subgroup A of index 2. Therefore, either $x \notin A$ or $y \notin A$. Since $(yx, x) = t$, one may assume that $x \notin A$ and $y \in A$. Then $xy \notin A$ and therefore $x^2, (xy)^2 \in Z(G)$. Furthermore, by statement 6 of Lemma 5.1, $t^x = t^{-1}$. Hence $(xy)^2 = t^{-1}x^2y^2$ and thus $t^{-1}y^2 \in Z(G)$. So, by statement 5 of Lemma 5.1, $t^{-4}y^8 = 1$. Thus $y^8 = t^4 \neq 1$, in a contradiction with 1.

3. $G'^2 \subseteq Z(G)$ is a consequence of 2 and statement 6 of Lemma 5.1. Furthermore for $t \in G'$, either $t^2 = 1$, or t has order 4 and $t \notin Z(G)$. Thus, again by statement 6 of Lemma 5.1, there is $x \in G$ such that $t^x = t^{-1}$, that is, $(x, t) = t^2$. Hence 3 follows.

4. That G_x is normal in G is a direct consequence of statement 6 in Lemma 5.1. Clearly, the natural image of x in G/G_x is central. Since G/G_x is a 2-group of Kleinian type, it follows from statement 5 of Lemma 5.1 that if G_x is properly contained in G' then $x^4 \in G_x$ as desired.

5. Let $x, y \in G$ and $t = (y, x)$. Clearly $(y, x^2) = (y, x)(y, x)^{x^{-1}} = tt^{x^{-1}}$. Because of statement 6 in Lemma 5.1, we also know that $t^{x^{-1}} = t$ or $t^{x^{-1}} = t^{-1}$. In the latter case we get that $(y, x^2) = 1$ and so (b) holds. In the former case $(t, x) = 1$ and $(y, x^2) = t^2$. If also $t^y = t$ then t is central in G, and thus, again by statement 6 in Lemma 5.1, $t^2 = 1$ and $(y, t) = t^2 = 1$. If, on the other hand, $t^y \neq t$, then, again by statement 6 in Lemma 5.1, $(y, t) = t^2$. So, in all cases we get that $(y, x^2) = t^2$. By part 2, we also know that $t^4 = 1$. Hence, $(y, tx^2) = (y, t)(y, x^2)^{t^{-1}} = (y, t)t^2 = 1$, as desired. ∎

The next three lemmas contain information on two and three generated 2-groups with a commutator of order 4.

Lemma 5.3 *Let $G = \langle x, y \rangle$ be a non-abelian 2-group and suppose $t = (y, x)$ has order 4. If G is of Kleinian type then one of the following conditions holds.*

1. $(x, t) = (y, t) = t^2$, $(xy, t) = 1$, $Z(G) = \langle t^2, x^2, y^2 \rangle$, *and one of the following conditions holds:*
   - *(a)* $t^2 = x^4y^4$*;*
   - *(b)* $t^2 \in \{x^2, y^2, x^4y^2, x^2y^4\}$*;*
   - *(c)* $x^4 = y^4 = 1$ *and* $t^2 = x^2y^2$*.*

2. $(x, t) = t^2$, $(y, t) = 1$, $Z(G) = \langle t^2, x^2, (xy)^2 \rangle = \langle t^2, x^2, ty^2 \rangle$, *and one of the following conditions holds:*
   - *(a)* $y^4 = 1$*;*
   - *(b) either* $ty^2 \in \{x^2, x^{-2}\}$ *or* $x^2 \in \{t^2, y^4\}$*;*
   - *(c)* $x^4 = 1$ *and* $t = y^{-2}$*.*

2' $(x, t) = 1$, $(y, t) = t^2$, $Z(G) = \langle t^2, y^2, (xy)^2 \rangle = \langle t^2, x^2, ty^2 \rangle$, *and one of the following conditions holds:*
   - *(a)* $x^4 = 1$*;*
   - *(b) either* $t^{-1}x^2 \in \{y^2, y^{-2}\}$ *or* $y^2 \in \{t^2, x^4\}$*;*
   - *(c)* $y^4 = 1$ *and* $t = x^2$*.*

Proof. Since by assumption, t has order 4, statement 6 of Lemma 5.1 yields that t is not central, $t^2 \in Z(G)$ and $\langle t \rangle$ is normal in G. Furthermore, $(x, t) = t^2$ or $(y, t) = t^2$. We deal with three mutually exclusive cases.

(1) First assume $(x,t)=(y,t)=t^2$. So $xt=t^{-1}x$ and $yt=t^{-1}y$. Hence $(y,x^2)=(y,x)(y,x)^{x^{-1}}=tt^{x^{-1}}=1$. Similarly $(x,y^2)=1$. Therefore $\langle t^2,x^2,y^2\rangle \subseteq Z(G)$. Since $t\not\in Z(G)$, and thus $t\not\in \langle t^2,x^2,y^2\rangle$, it follows that $G/\langle t^2,x^2,y^2\rangle$ is a non-abelian group of order 8. Hence we obtain that $Z(G)=\langle t^2,x^2,y^2\rangle$. Also $(xy,t)=1$ because $(xy,t)=(y,t)^{x^{-1}}(x,t)=(t^2)^{x^{-1}}t^2=1$.

Let $H=\langle x^2,y^2\rangle$, a central subgroup of G. Note that $(xy)^2=t^3x^2y^2$ and thus, in the group G/H, one has that $\overline{t}^{-1}=(\overline{xy})^2$ and $(\overline{xy})^{\overline{y}}=(\overline{xy})^{-1}$. By statement 1 of Lemma 5.2 we know that $(xy)^8=1$. Hence, there is an epimorphism $f:D_{16}\to G/H$ given by $f(a)=\overline{xy}$ and $f(b)=\overline{y}$, where $D_{16}=\langle a\rangle_8\rtimes\langle b\rangle_2$ is the dihedral group of order 16. Because of statement 3 in Lemma 5.1, D_{16} is not of Kleinian type. However, by Lemma 4.2, G/H is of Kleinian type. Hence, $\ker f\neq 1$ and therefore the order of G/H divides 8. Thus

$$1\neq t^2\in\langle x^2,y^2\rangle. \tag{4}$$

We consider three cases: (i) $x^4\neq 1$, (ii) $y^4\neq 1$ and (iii) $x^4=y^4=1$.

(i) Suppose $x^4\neq 1$. So, by statement 1 of Lemma 5.2, x^4 has order 2 and, by the above, x^2 is central in G. Then $e=\frac{1}{2}(1-t^2)\ \frac{1}{2}(1-x^4)$ is a nonzero central idempotent of $\mathbb{Q}G$. Clearly, the semi-simple $\mathbb{Q}$-algebra $A=\mathbb{Q}Ge$ is contained in $\mathbb{Q}G(1-\widehat{t})$. The latter, and thus also A, is a direct sum of non-commutative simple algebras (see the beginning of the proof of Lemma 5.2).

Let $f=\widehat{x^2y^2}e$, a central idempotent of A. $f=0$. Because $\{xy,txy\}$ is a full conjugacy class of G, we get that $z=(1+t)xyf$ and $i=x^2f$ are central elements of Af. Since $x^4e=-e$, $t^2e=-e$, $x^2y^2f=f$ and $ef=f$, we get that $i^2=-f$ and $z^2=(1+t)^2(xy)^2f=(1+2t+t^2)t^{-1}x^2y^2=2f$. If $f\neq 0$ then there exists a primitive central idempotent f_1 of Af such that $i^2f_1=-f_1$ and $z^2f_1=2f_1$. Then $\mathbb{Q}Gf_1$ is a non-commutative simple quotient of $\mathbb{Q}G$ having a central subfield isomorphic to $\mathbb{Q}(\sqrt{2},i)=\mathbb{Q}(\xi_8)$, contradicting the last statement of Proposition 3.2. Thus $f=0$ and this implies that $\langle x^2y^2\rangle\cup t^2x^4\langle x^2y^2\rangle=t^2\langle x^2y^2\rangle\cup x^4\langle x^2y^2\rangle$. Therefore

$$t^2\in\langle x^2y^2\rangle \text{ or } x^4\in\langle x^2y^2\rangle. \tag{5}$$

If $x^4=y^4$ then $(x^2y^2)^2=x^8=1$, and it follows that $t^2=x^2y^2$ or $x^2=y^2$. In both situations the central elements $i=x^2e$ and $z=(1+t)xye$ of A are such that $i^2=-e$ and $z^2=-2e$. Hence A has a central subfield isomorphic with $\mathbb{Q}(i,\sqrt{-2})=\mathbb{Q}(\xi_8)$, again yielding a contradiction. Thus $x^4\neq y^4$. Since, by statement 1 of Lemma 5.2, $x^8=y^8=1$, we obtain that x^2y^2 has order 4. As also both t and x^2 have order 4, (5) then implies that either $t^2=x^4y^4$ or $y^4=1$. In the first case 1(a) holds. In the second case $y^4=1$ and thus (4) implies that $t^2\in\{x^4,y^2,x^4y^2\}$. If $t^2=x^4$ then 1(a) holds, otherwise 1(b) holds.

(ii) Suppose $y^4\neq 1$. By symmetry with case (i), we also obtain that either 1(a) or 1(b) holds.

(iii) Suppose $x^4=y^4=1$. By (4), we clearly get that $x^2\neq 1$ or $y^2\neq 1$. If $y^2\in\langle x^2\rangle$ then, by (4), we obtain that $1\neq t^2\in\langle x^2\rangle$ and thus $t^2=x^2$; hence 1(b) holds. Similarly, if $x^2\in\langle y^2\rangle$ then $t^2=y^2$ and again 1(b) holds. Otherwise $\langle x^2,y^2\rangle\cong C_2\times C_2$ and thus by (4) one of the following holds: $t^2=y^2$, $t^2=x^2$ or $t^2=x^2y^2$. Hence 1(b) or 1(c) holds. This finishes the proof when $(x,t)=(y,t)=t^2$.

(2) Second assume that $(y,t)=1$. Hence, since t is not central, $(x,t)=t^2$. Set $x_1=x$, $y_1=yx$ and $t_1=(y_1,x_1)$. Then $t_1=t$ and $(x_1,t_1)=(y_1,t_1)=t_1^2$. Thus, by (1), x_1 and y_1 satisfy one of the conditions of (1). In particular $Z(G)=\langle t^2,x_1^2,y_1^2\rangle$. Notice that $y_1^2=t^{-1}x^2y^2$ and $y_1^4=t^2x^4y^4$. Then $Z(G)=\langle t^2,x_1^2,y_1^2\rangle=\langle t^2,x^2,(xy)^2\rangle=\langle t^2,x^2,ty^2\rangle$.

Thus if x_1 and y_1 satisfy 1(a) then $y^4=1$, that is, condition 2(a) holds.

Next assume that x_1 and y_1 satisfy 1(b). If $t_1^2=x_1^2$ then $t^2=x^2$. If $t_1^2=y_1^2$ then $t=x^{-2}y^{-2}$. If $t_1^2=x_1^4y_1^2$ then $t=x^2y^{-2}$. If $t_1^2=x^2y_1^4$ then $x^2=y^4$. So always 2(b) holds.

Finally assume that x_1 and y_1 satisfy 1(c). Then $x^4 = 1$ and $t = y^{-2}$, that is 2(c) holds.

(3) Third assume that $(x,t) = 1$ and therefore $(y,t) = t^2$. Setting $x_1 = y$ and $y_1 = x$, one has $t_1 = (y_1, x_1) = t^{-1}$, $(y_1, t_1) = 1$ and $(x_1, t) = t_1^2$. Therefore x_1, y_1 and t_1 satisfy one of the conditions of 2 and this is equivalent with x, y and t satisfying one of the conditions of 2'. ∎

We will need the following remark.

Remark 5.4 It follows from the proof of Lemma 5.3 that if G is a non-abelian 2-group of Kleinian type with a commutator t of order 4 then there exist $x_1, y_1, x_2, y_2, x_2', y_2' \in G$ with $t = (y_1, x_1) = (y_2, x_2) = (y_2', x_2')$, and so that x_1 and y_1 satisfy condition 1, x_2 and y_2 satisfy condition 2, and x_2' and y_2' satisfy condition 2'. Moreover if x_1 and y_1 satisfy 1(a) (respectively, 1(b) or 1(c)) then x_2 and y_2 satisfy 2(a) (respectively, 2(b) or 2(c)) and x_2' and y_2' satisfy 2'(a) (respectively, 2'(b) or 2'(c)).

Lemma 5.5 *Let $G = \langle x_1, x_2, x_3 \rangle$ be a non-abelian 2-group such that $x_2^4 \neq 1$, $x_3 \in Z(G)$, $(x_1, t) = t^2 \neq 1$ and $(x_2, t) = 1$ with $t = (x_2, x_1)$. If G is of Kleinian type then $x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$.*

Proof. We argue by contradiction. So suppose $G = \langle x_1, x_2, x_3 \rangle$ has minimal order among the possible counterexamples to the lemma. In particular $x_3^2 \neq 1$. By statement 5 of Lemma 5.1, x_3 has order 4. Let $G_1 = \langle x_1, x_2 \rangle$. By Lemma 5.3, $Z(G_1) = \langle x_1^2, t^2, tx_2^2 \rangle$ and therefore $Z(G_1)^2 = \langle x_1^4, t^2 x_2^4 \rangle$.

Suppose $z \in Z(G_1)$ is such that $1 \neq z$, $t^2 \notin \langle z \rangle$ and $x_2^4 \notin \langle z \rangle$. Then $G_1' \cap \langle z \rangle = \langle t \rangle \cap \langle z \rangle = 1$ and the minimality of the order of $|G|$ applied to the group $G/\langle z \rangle$ yields $x_3^2 \in \langle z, Z(G_1)^2 \rangle$.

If $t^2 \neq x_2^4$ then $z = t^3 x_2^2$ is a non trivial central element of order 4 so that $t^2 \notin \langle z \rangle$ and $x_2^4 \notin \langle z \rangle$. Hence, by the previous, x_3^2 is an element of order 2 of the group $H = \langle t^3 x_2^2, Z(G_1)^2 \rangle$. Since $t^3 x_2^2 \in Z(G_1)$, one has that $H = Z(G_1)^2 \cup t^3 x_2^2 Z(G_1)^2$. Then $x_3^2 = t^3 x_2^2 w^2$ for some $w \in Z(G_1)$. So $1 = x_3^4 = (t^3 x_2^2)^2 = t^2 x_2^4 \neq 1$, a contradiction.

Thus we have that $t^2 = x_2^4 \neq 1$. Lemma 5.3 therefore implies that we have one of the following properties: (i) $x_1^2 = t^2 = x_2^4$ or $t = x_1^{\pm 2} x_2^{-2}$ (this is case 2(b)), or (ii) $x_1^4 = 1$ and $t = x_2^{-2}$ (this is case 2(c)). In both cases we have $x_1^4 = 1$ and therefore $Z(G_1)^2 = 1$. Thus, if $z \in Z(G_1)$ has order 2 and $z \neq t^2 = x_2^4$ then, by the above argument, we have that $x_3^2 \in \langle z, Z(G_1)^2 \rangle = \langle z \rangle$; hence $x_3^2 = z$. This shows that x_3^2 is the unique central element of order 2 in $Z(G)$. Therefore $Z(G)$ is cyclic generated by x_3, $Z(G_1) = \langle t^2 \rangle$ and $x_3^3 = t^2$. Since tx_2^2 is central of order at most 2 we thus get that either $tx_2^2 = 1$ or $tx_2^2 = t^2$, that is, $t = x_2^{\pm 2}$.

Then $K = \langle x_2, x_3 \rangle$ is an abelian subgroup of index 2 in G. Let $H = \langle tx_3^{-1} \rangle$. Clearly K/H is cyclic (generated by $\overline{x_2}$). Thus $K = N_G(H)$ and (K, H) is a strong Shoda pair of G. Using also statement 2 of Lemma 5.1, it follows that $[K : H] \leq 4$ and hence $t^2 = x_2^4 \in H = \{1, tx_3^{-1}\}$. Then $t = x_3^{-1} \in Z(G)$, a contradiction. ∎

Lemma 5.6 *Let $G = \langle x_1, x_2, x_3 \rangle$ be a 2-group of Kleinian type with $G' = \langle t \rangle$ of order 4. Let $t_{ij} = (x_j, x_i)$ with $1 \leq i < j \leq 3$. Assume that $t = t_{12}$, $(x_1, t) = t^2$, $(x_2, t) = 1$ and $t_{23} = 1$.*

1. *If $t_{13} \in \langle t^2 \rangle$ then $x_3^4 = 1$. If, moreover, $x_2^4 \neq 1$ then either $t_{13} = 1$ and $x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$ or $t_{13} = t^2$ and $x_2^4 x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$.*

2. *If $t_{13} \notin \langle t^2 \rangle$ then $x_3^4 = x_2^4$. If, moreover, $x_2^4 \neq 1$ then either $t_{13} = t^{-1}$ and $x_2^2 x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$ or $t_{13} = t$ and $t^2 x_2^2 x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$.*

Proof. 1. Assume first that $t_{13} \in \langle t^2 \rangle$. If $t_{13} = 1$ then $x_3 \in Z(G)$ and if $t_{13} = t^2$ then $x_2^2 x_3 \in Z(G)$. In both cases, because of statement 5 of Lemma 5.1 and statement 1 of Lemma 5.2, we obtain that $x_3^4 = 1$. The second statement is a consequence of Lemma 5.5, applied to $\langle x_1, x_2, x_3 \rangle$ if $t_{13} = 1$, and to $\langle x_1, x_2, x_2^2 x_3 \rangle$ if $t_{13} = t^2$.

2. Assume second that $t_{13} \notin \langle t^2 \rangle$. Then $t_{13} \in \{t, t^{-1}\}$. If $t_{13} = t^{-1}$ then $x_2 x_3 \in Z(G)$ and hence $(x_2 x_3)^4 = x_2^4 x_3^4 = 1$. So $x_2^4 = x_3^4$. If, moreover, $x_2^4 \neq 1$ then by Lemma 5.5 we have that $x_2^2 x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$. On the other hand if $t_{13} = t$ then $t x_2 x_3 \in Z(G)$ and hence $(t x_2 x_3)^4 = x_2^4 x_3^4 = 1$. So $x_2^4 = x_3^4$. If, moreover, $x_2^4 \neq 1$, then again by Lemma 5.5, we have that $t^2 x_2^2 x_3^2 \in Z(\langle x_1, x_2 \rangle)^2$. This finishes the proof. ∎

We need one more lemma before giving the proof of (D) implies (F) for nilpotent groups.

Lemma 5.7 *Let G be a finite non-abelian 2-group of Kleinian type. Assume $G' \subseteq Z(G)$ and $Z(G/T)$ has exponent 2 for every proper subgroup T of G'. Then G is an epimorphic image of either $C_2^n \times \mathcal{W}$, $\mathcal{W}_{1n}$ or $\mathcal{W}_{2n}$ for some n.*

Proof. Applying the assumptions for $T = 1$ one deduces that $Z(G)$ and G' have exponent 2. Then $G/Z(G)$ has exponent 2 and therefore G has exponent 4.

First, we prove that G has an abelian subgroup of index 2. Otherwise, by statement 1 of Lemma 5.1, $G/Z(G) = \langle \overline{x_1} \rangle_2 \times \langle \overline{x_2} \rangle_2 \times \langle \overline{x_3} \rangle_2$ for some $x_1, x_2, x_3 \in G$. Hence $G = \langle Z(G), x_1, x_2, x_3 \rangle$ and $G' = \langle t_{12} \rangle_2 \times \langle t_{13} \rangle_2 \times \langle t_{23} \rangle_2$, where $t_{i,j} = (x_j, x_i)$ for $1 \leq i < j \leq 3$. If $x \in G$ then $G_x = \langle (x, y) : y \in G \rangle$ is a proper subgroup of G' and the image of x in G/G_x is central. Therefore $x^2 \in G_x$, by assumption. In particular, $x_1^2 = t_{12}^{\alpha_2} t_{13}^{\alpha_3}$, $x_2^2 = t_{12}^{\beta_1} t_{23}^{\beta_3}$ and $x_3^2 = t_{13}^{\gamma_1} t_{23}^{\gamma_2}$, for some α_2, α_3, β_1, β_3, γ_1, $\gamma_2 \in \{0, 1\}$. Then $(x_1 x_2)^2 = t_{12}^{1+\alpha_2+\beta_1} t_{13}^{\alpha_3} t_{23}^{\beta_3} \in \langle t_{12}, t_{13} t_{23} \rangle$, $(x_1 x_3)^2 = t_{12}^{\alpha_2} t_{13}^{1+\alpha_3+\gamma_1} t_{23}^{\gamma_2} \in \langle t_{13}, t_{12} t_{23} \rangle$ and $(x_2 x_3)^2 = t_{12}^{\beta_1} t_{13}^{\gamma_1} t_{23}^{1+\beta_3+\gamma_2} \in \langle t_{23}, t_{12} t_{13} \rangle$. This implies that the α's, β's and γ's with the same subindex are equal, so $x_1^2 = t_{12}^{a_2} t_{13}^{a_3}$, $x_2^2 = t_{12}^{a_1} t_{23}^{a_3}$ and $x_3^2 = t_{13}^{a_1} t_{23}^{a_2}$, for some $a_1, a_2, a_3 \in \{0, 1\}$. Applying once more the property to $x_1 x_2 x_3$ one obtains that

$$t_{12}^{1+a_1+a_2} t_{13}^{1+a_1+a_3} t_{23}^{1+a_2+a_3} = t_{12} t_{13} t_{23} x_1^2 x_2^2 x_3^2 = (x_1 x_2 x_3)^2 \in \langle t_{12} t_{13}, t_{12} t_{23} \rangle$$

and therefore $3 + 2a_1 + 2a_2 + 2a_3 \equiv 0 \mod 2$, a contradiction.

Therefore $G = \langle x, y_1, \ldots, y_n \rangle$ where $Y = \langle y_1, \ldots, y_n \rangle$ is an abelian subgroup of index 2 in G. In particular $G' \subseteq \langle y_1, \ldots, y_n \rangle$. If $y_i^2 = 1$ for every $i = 1, \ldots, n$, then G is an epimorphic image of $\mathcal{W}_{1n}$, as desired. Otherwise, we may assume without loss of generality that y_1 has exponent 4 and so $(y_1, x) \neq 1$, because $Z(G)$ has exponent 2. If $|G'| = 2$ then $(y_i, x) \in \langle (y_1, x) \rangle$ and, replacing y_i by $y_1 y_i$ if needed, one may assume that $y_i \in Z(G)$ for every $i \geq 2$. Then G is a quotient of $\mathcal{W} \times C_2^{n-1}$. Finally suppose that $|G'| > 2$. Then, for every $i \geq 2$, replacing y_i by $y_1 y_i$ if needed, we may assume that y_i has order 4. Then $G_{y_i} = \langle (y_i, x) \rangle$ is a proper subgroup of G' and therefore $1 \neq y_i^2 \in G_{y_i} = \langle (y_i, x) \rangle$ and so $y_i^2 = (y_i, x)$. It follows that G is an epimorphic image of $\mathcal{W}_{2n}$. ∎

We are ready to prove (D) implies (F) for nilpotent groups. So let G be a non-abelian finite nilpotent group of Kleinian type. Hence by statements 4 and 5 of Lemma 5.1 $G = G_3 \times G_2$, where G_3 is an elementary abelian 3-group, G_2 is a non-abelian 2-group and the exponent of $Z(G) = G_3 \times Z(G_2)$ divides 4 or 6.

We will deal separately with three cases. (1) $G_3 \neq 1$, (2) $G_3 = 1$ and $G' \subseteq Z(G)$ and (3) $G_3 = 1$ and $G' \not\subseteq Z(G)$.

(1) Assume G_3 is not trivial. We will show that G_2 satisfies the hypothesis of Lemma 5.7. This implies that G_2 is isomorphic to a quotient of either $\mathcal{W} \times C_2^n$, $\mathcal{W}_{1n}$ or $\mathcal{W}_{2n}$, for some n. Hence condition (F.1) of Theorem 1 holds.

If T is a proper subgroup of G_2' then, since also G/T is of Kleinian type, the exponent of $Z(G/T)$ is 6, by statement 5 of Lemma 5.1. Hence $Z(G_2/T)$ has exponent 2, as desired.

Next we need to show that $G' \subseteq Z(G)$. We prove this by contradiction. So assume that $G' \not\subseteq Z(G)$. Then, by statement 6 of Lemma 5.1 and statement 4 of Lemma 5.2, there exist $x, y \in G_2$ such that $t = (y, x)$ has order 4. Because of Remark 5.4 one may assume without loss of generality that x and y satisfy condition 1 of Lemma 5.3. So $x^2, y^2 \in Z(G_2)$ and therefore $x^4 = y^4 = 1$. Since $t^2 \neq 1$, case 1(a) does not hold and so $t^2 \in \{x^2, y^2, x^2y^2\}$. By symmetry one may assume that $t^2 = x^2$ or $t^2 = x^2y^2$. Notice that $(xy)^2 = t^{-1}x^2y^2$ and therefore xy has order 8. Thus $H = \langle x, y\rangle$ is a non-abelian group of exponent 8 which is an epimorphic image of one of the following two groups:

$$\begin{array}{rcl} H_1 & = & \langle a, b | a^4 = b^4 = 1, t = (b, a), t^a = t^b = t^{-1}, t^2 = a^2\rangle \\ H_2 & = & \langle a, b | a^4 = b^4 = 1, t = (b, a), t^a = t^b = t^{-1}, t^2 = a^2b^2\rangle \end{array}$$

On the other hand, $\mathbb{Q}(C_3 \times Q_{16})$ has an epimorphic image isomorphic to $\mathbb{Q}(\xi_3) \otimes_{\mathbb{Q}} \mathbb{H}(\mathbb{Q}(\sqrt{2})) \cong M_2(\mathbb{Q}(\xi_3, \sqrt{2}))$ and $\mathbb{Q}(C_3 \times D_{16}^-)$ has an epimorphic image isomorphic to $\mathbb{Q}(\xi_3) \otimes_{\mathbb{Q}} M_2(\mathbb{Q}(\sqrt{-2})) \cong M_2(\mathbb{Q}(\xi_3, \sqrt{-2}))$ (see (3)). Then, statement 2 of Lemma 5.1 implies that neither $C_3 \times Q_{16}$ nor $C_3 \times D_{16}^-$ are of Kleinian type. Since $H_1/\langle a^2b^2\rangle \cong Q_{16}$ and $H_2/\langle b^2\rangle \cong D_{16}^-$, Lemma 4.2 implies that neither $C_3 \times H_1$ nor $C_3 \times H_2$ are of Kleinian type. Since $|H| \geq 16$ and $|H_1| = |H_2| = 32$, we have that H is a non-abelian group of order 16 with an element of order 8. This implies that H is isomorphic to either $D_{16}, D_{16}^+, D_{16}^-$ or Q_{16}. However H is not isomorphic to D_{16} because the latter is not of Kleinian type, and it is also not isomorphic to D_{16}^+ because the commutator of D_{16}^+ has order 2. Moreover the same argument as above shows that H is not isomorphic to neither Q_{16} nor D_{16}^-. This yields in all cases a contradiction. So $G' \subseteq Z(G)$ and this finishes the proof of (1).

(2) Assume that $G_3 = 1$ and $G' \subseteq Z(G)$. We prove that G is isomorphic to a quotient of either $\mathcal{V} \times A$, $\mathcal{U}_1 \times A$, $\mathcal{U}_2 \times A$, $\mathcal{V}_{1n}$ or $\mathcal{V}_{2n}$, for an abelian group A of exponent 4. Hence condition (F.2) of Theorem 1 holds.

From statement 1 of Lemma 5.2 and statement 5 of Lemma 5.1, we know that the exponent of G divides 8 and the exponent of $Z(G)$ divides 4. Moreover, the assumptions and statement 6 of Lemma 5.1 imply that G' is of exponent 2. Hence $g^2 \in Z(G)$ for all $g \in G$.

If G does not contain an abelian subgroup of index 2 then, by statement 1 of Lemma 5.1, $G = \langle Z(G), y_1, y_2, y_3\rangle$, and $G/Z(G)$ and $G' = \langle t_{ij} = (y_j, y_i) \mid 1 \leq i < j \leq 3\rangle$ are both elementary abelian groups of order 8. By statement 4 of Lemma 5.2, it follows that there exist α_2, α_3, β_1, β_3, γ_1 and γ_2 in $\{0, 1\}$ such that $y_1^4 = t_{12}^{\alpha_2}t_{13}^{\alpha_3}$, $y_2^4 = t_{12}^{\beta_1}t_{23}^{\beta_3}$ and $y_3^4 = t_{13}^{\gamma_1}t_{23}^{\gamma_2}$. Applying again part 4 of Lemma 5.2, it follows that $(y_1y_2)^4 = t_{12}^{\alpha_2+\beta_1}t_{13}^{\alpha_3}t_{23}^{\beta_3} \in \langle t_{12}, t_{13}t_{23}\rangle$, $(y_1y_3)^4 = t_{12}^{\alpha_2}t_{13}^{\alpha_3+\gamma_1}t_{23}^{\gamma_2} \in \langle t_{13}, t_{12}t_{23}\rangle$ and $(y_2y_3)^4 = t_{12}^{\beta_1}t_{13}^{\gamma_1}t_{23}^{\beta_3+\gamma_2} \in \langle t_{23}, t_{12}t_{13}\rangle$. Hence $\alpha_3 = \beta_3$, $\alpha_2 = \gamma_2$ and $\beta_1 = \gamma_1$. To simplify notation, put $a_1 = \beta_1$, $a_2 = \alpha_2$ and $a_3 = \alpha_3$. Then, once more applying statement 4 of Lemma 5.2, we get

$$\begin{array}{rcl} y_1^4 & = & t_{12}^{a_2}t_{13}^{a_3} \\ y_2^4 & = & t_{12}^{a_1}t_{23}^{a_3} \\ y_3^4 & = & t_{13}^{a_1}t_{23}^{a_2} \end{array} \quad \text{and} \quad \begin{array}{rcl} (y_1y_2)^4 & = & t_{12}^{a_1+a_2}t_{13}^{a_3}t_{23}^{a_3} \\ (y_1y_3)^4 & = & t_{12}^{a_2}t_{13}^{a_1+a_3}t_{23}^{a_2} \\ (y_2y_3)^4 & = & t_{12}^{a_1}t_{13}^{a_1}t_{23}^{a_2+a_3} \\ (y_1y_2y_3)^4 & = & t_{12}^{a_1+a_2}t_{13}^{a_1+a_3}t_{23}^{a_2+a_3}. \end{array} \tag{6}$$

Because each $a_i \in \{0, 1\}$, we obtain that at least one of the seven elements in (6) is equal to 1. Without loss of generality, we may assume that $y_1^4 = 1$, and hence $a_2 = a_3 = 0$. Then $y_2^4 = t_{12}^{a_1}$ and $y_3^4 = t_{13}^{a_1}$. If $a_1 = 0$ then it follows that G is an epimorphic image of $\mathcal{U}_1 \times C_4^n$ for some n. If $a_1 = 1$ then G is an epimorphic image of $\mathcal{U}_2 \times C_4^n$ for some n.

We now consider the case that G has an abelian subgroup $\langle y_1, \ldots, y_n\rangle$ of index 2. Write $G = \langle x, y_1, \ldots, y_n\rangle$. If $y_i^4 = 1$ for every i, then G is an epimorphic image of $\mathcal{V}_{1n}$. So assume that

some y_i, say y_1, has order 8. In particular $(y_1, x) \neq 1$. As in the case where $G_3 \neq 1$ and G_2 has an abelian subgroup of index 2, if $|G'| = 2$ then one may assume that y_i is central for every $i \geq 2$ and therefore $y_i^4 = 1$. This implies that G is an epimorphic image of $\mathcal{V} \times C_4^{n-1}$. Finally, assume that y_1 has order 8 and $|G'| > 2$. Again following the same pattern as in the case of $G_3 \neq 1$, replacing y_i by $y_1 y_i$ one may assume that each y_i has order 8 and applying statement 4 of Lemma 5.2, one deduces that $y_i^4 = (y_i, x)$ for every i. It follows that G is an epimorphic image of $\mathcal{V}_{2n}$.

(3) Assume that $G_3 = 1$ and $G' \not\subseteq Z(G)$. We prove that G is an epimorphic image of either $\mathcal{T} \times A$, $\mathcal{T}_{1n}$, $\mathcal{T}_{2n}$ or $\mathcal{T}_{3n}$ for A an elementary abelian 2-group.

By statement 1 of Lemma 5.2, the exponent of G divides 8. By statements 1 and 6 of Lemma 5.1, G has an abelian subgroup $Y = \langle y_1, \dots, y_n \rangle$ of index 2 and G' has exponent 4. Then $G = \langle Y, x \rangle$ for some $x \in G$ and $G' = \langle t_1, \dots, t_n \rangle$, where $t_i = (y_i, x)$ for $i = 1, \dots, n$. We may assume, without loss of generality, that t_1 is of order 4 (and thus t_1 is not central). Since $G' \subseteq Y$, $(t_i, y_j) = 1$ for all $1 \leq i, j \leq n$. If t_j is not central then, by Lemma 5.3, $(x, t_j) = t_j^2$. If t_j is central, and thus of order two, we also get that $(x, t_j) = 1 = t_j^2$. So, in all cases we have $(x, t_j) = t_j^2$.

We now show that we may assume that $\langle t_1 \rangle \cap \langle t_i \rangle = 1$ for every $i \geq 2$.

Because the order of t_i divides 4, this is clear if $t_i^2 \notin \langle t_1 \rangle$. If $t_i \in \langle t_1 \rangle$, say $t_i = t_1^a$ then we replace y_i by $y_i' = y_1^{-a} y_i$ to make $(y_i', x) = 1$, because $(y_i', x) = (y_1^{-a} y_i, x) = (y_i, x)^{y_1^a} (y_1^{-a}, x) = t_i t_1^{-a} = 1$. In the remaining case $t_i \notin \langle t_1 \rangle$ and $t_i^2 \in \langle t_1^2 \rangle$. Then either $t_i^2 = 1$ or $t_i^2 = t_1^2$. If $t_i^2 = 1$ then the claim is clear. If $t_i^2 = t_1^2$ then replacing y_i by $y_i' = y_1 y_i$ we obtain that $(y_i', x) = (y_1 y_i, x) = t_i^{y_1^{-1}} t_1 = t_i t_1 \notin \langle t_1 \rangle$ and $(y_i', x)^2 = 1$, which finishes the proof of the claim. So from now on we assume that for $i \geq 2$, $\langle t_1 \rangle \cap \langle t_i \rangle = 1$. Since the order of t_i divides 4 and the order of t_1 is 4, this implies that $\langle t_i \rangle \cap \langle t_1 t_i \rangle = 1$ for $i \geq 2$.

For $i = 1, \dots, n$ we put $F_i = \langle x, y_1, y_i \rangle$ and we prove three claims.

Claim 1: If $y_1^4 = 1$ then $y_j^4 = 1$ for every j (with $1 \leq j \leq n$).

Indeed, suppose $y_1^4 = 1$. If $t_j = 1$ then y_j is central in G and thus, by statement 5 of Lemma 5.1, we get at once that $y_j^4 = 1$. So assume that $t_j \neq 1$. We now apply statement 4 of Lemma 5.2 to the group F_j. Since $\langle t_1 \rangle \cap \langle t_j \rangle = 1$, one has $(F_j)_{y_j} = \langle t_j \rangle \neq F_j' \neq \langle y_1 y_j \rangle = (F_j)_{y_1 y_j}$ and hence $y_j^4 \in \langle t_j \rangle$ and $y_j^4 = (t_1 t_j)^4 \in \langle t_1 t_j \rangle$. Thus $y_j^4 \in \langle t_j \rangle \cap \langle t_1 t_j \rangle = 1$. This proves Claim 1.

Claim 2: If $x^4 = 1$ and $t_1 = y_1^{-2}$ then $t_j = y_j^{-2}$ for every j (with $1 \leq j \leq n$).

Indeed, suppose $x^4 = 1$ and $t_1 = y_1^{-2}$. Let $Z = \langle x^2, t_1^2 \rangle$. Then Z is a subgroup of $Z(F_1)$. Moreover $y_1^{-2} = t_1 \notin Z(F_1)$ and $(x y_1^i, t_1) = t_1^2 \neq 1$ for every i. This shows that $Z = Z(F_1)$. Hence $Z(F_1)^2 = 1$.

Let j be such that $2 \leq j \leq n$. Since $t_1^2 \notin \langle t_j \rangle$ (because $\langle t_1 \rangle \cap \langle t_j \rangle = 1$ and t_1 has order 4) we can apply Lemma 5.5 to the elements $x_1 = \overline{x}$, $x_2 = \overline{y_1}$ and $x_3 = \overline{y_j}$ of the non-abelian Kleinian group $F_j / \langle t_j \rangle$ and deduce that $\overline{y_j}^2 \in \overline{Z}^2 = 1$. Hence

$$y_j^2 \in \langle t_j \rangle \tag{7}$$

We now proceed by considering the possible orders of t_j. If $t_j = 1$ then (7) implies that $y_j^{-2} = 1 = t_j$, as desired. If t_j has order 4 then, again because $\langle t_1 \rangle \cap \langle t_j \rangle = 1$, the second part of Lemma 5.6 is applicable to the group $F_j / \langle t_1^{-1} t_j \rangle$, for $x_1 = \overline{x}$, $x_2 = \overline{y_1}$ and $x_3 = \overline{y_j}$. It follows that $\overline{t_1^2 y_1^2 y_j^2} \in \overline{Z}^2 = 1$. Hence $t_1^2 y_1^2 y_j^2 = t_1 y_j^2 \in \langle t_1^{-1} t_j \rangle$. Combining this with (7), we obtain $y_j^2 \in \langle t_j \rangle \cap t_1^{-1} \langle t_1^{-1} t_j \rangle = \{t_j^{-1}\}$, as desired. If t_j has order two then again (7) implies that either $y_j^2 = t_j$ or $y_j^2 = 1$. The former is as desired. In the second case we can apply Lemma 5.5 to the non-abelian Kleinian group $F_j / \langle t_1^2 t_j \rangle$ (note that $\overline{t_1 y_j}$ is central in this group). It follows that $(\overline{t_1 y_j})^2 \in \overline{Z^2} = 1$ and thus $t_1^2 = t_1^2 y_j^2 \in \langle t_1^2 t_j \rangle$. Hence $t_1^2 = t_1^2 t_j$, a contradiction. This finishes the proof of Claim 2.

Claim 3: If G' is not cyclic then $y_i^4 \in \langle t_i \rangle$ for every i with $1 \le i \le n$. If, furthermore, $t_i^2 = y_i^4 \neq 1$ for some $i \ge 1$, then $x^4 = 1$.

Assume that G' is not cyclic. Then $G_{y_i} = \langle t_i \rangle \neq G'$, for each $i \ge 1$. Hence, by statement 4 of Lemma 5.2, $y_i^4 \in \langle t_i \rangle$, as desired. Assume, furthermore, that $x^4 \neq 1$ and $t_i^2 = y_i^4 \neq 1$ for some $i \ge 1$. By Lemma 5.3, $t_i y_i^2 = x^{\pm 2}$ and therefore $1 = t_i^2 y_i^4 = x^4$, a contradiction. Hence the claim follows.

We now consider 3 cases.

Case 1. Suppose $y_1^4 = 1$.

Because of Claim 1 we obtain that $y_j^4 = 1$ for every j. Hence we conclude that G is a quotient of $\mathcal{T}_{1n}$.

Case 2. Suppose $x^4 = 1$ and $t_1 = y_1^{-2}$.

Because of Claim 2 we conclude that G is a quotient of $\mathcal{T}_{2n}$.

Case 3. Suppose that neither Case 1 nor Case 2 hold.

Claim 4. One may assume that, for every $i \ge 1$, if $t_i^2 \neq 1$ then one has $y_i^4 \neq 1$, $t_i \neq y_i^{-2}$ and either $t_i y_i^2 = x^{\pm 2}$ or $x^2 \in \{t_i^2, y_i^4\}$.

Suppose that $t_i^2 \neq 1$. Then x and y_i satisfy condition 2 of Lemma 5.3 and hence one of the three cases (a), (b) or (c) of this statement holds. If $y_i^4 = 1$ then interchanging the roles of y_1 and y_i one may assume that Case 1 holds. So one may assume that $y_i^4 \neq 1$ and hence (a) does not hold. Suppose now that $t_i = y_i^{-2}$. If $x^4 \neq 1$ then (c) does not hold and from (b) we get that $1 = t_i y_i^2 = x^{\pm 2} \neq 1$, a contradiction. Thus in this case $x^4 = 1$ and interchanging again the roles of y_1 and y_i one may assume that Case 2 holds. So one may assume that $y_i^4 \neq 1$ and $t_i \neq y_i^{-2}$. Then neither (a) nor (c) holds. Thus (b) holds and this finishes the proof of Claim 4.

Assume that G' is cyclic. Thus $\langle t_i \rangle \subseteq \langle t_1 \rangle$. Since we already know that $\langle t_1 \rangle \cap \langle t_i \rangle = 1$, for $i \ge 2$, this implies that $t_i = 1$. Hence y_i is central for $i \ge 2$. By Lemma 5.5, we get that $y_i^2 = z_i^2$ for some $z_i \in Z(F_1)$. Then, replacing y_i by $y_i z_i$, we may assume that $y_i^2 = 1$, for $i \ge 2$. Thus G is an epimorphic image of $F_1 \times C_2^{n-1}$. We are going to show that F_1 is an epimorphic image of $\mathcal{T}$ and therefore G is an epimorphic image of $\mathcal{T} \times C_2^{n-1}$.

First assume that $x^4 \neq 1$ and so, by Claim 4, $t_1 y_i^2 = x^{\pm 2}$. If $t_1 y_1^2 = x^{-2}$ then $(xy_1)^2 = t_1^{-1} x^2 y_1^2 = x^4 y_1^4 = t_1^2 = (xy_1, t_1)$ and $t_1 = (y_1, xy_1)$. Then replacing x by xy_1 one sees that F_1 is an epimorphic image of $\mathcal{T}$. A similar computation shows that if $t_1 y_1^2 = x^2$ then replacing x by xy_1^{-1} one deduces that F_1 is an epimorphic image of $\mathcal{T}$. Second assume that $x^4 = 1$. By Claim 4, either $t_1 y_1^2 = x^2$, $x^2 = y_1^4$ or $x^2 = t_1^2$. If $x^2 = t_1^2$ then H is clearly an epimorphic image of $\mathcal{T}$. If $x^2 = y_1^4$ then $x \mapsto xy_1^2$ and $y \mapsto y_1$ induces an epimorphism $\mathcal{T} \to H$ and if $t_1 y_1^2 = x^2$ one gets an epimorphism $\mathcal{T} \to H$ given by $x \mapsto xy_1$ and $y \mapsto y_1$. This finishes the proof if G' is cyclic.

Assume that G' is not cyclic.

By Claim 3, $y_i^4 \in \langle t_i \rangle \cap Z(G)$ for every i. In particular, if $t_i^2 \neq 1$ (for example for $i = 1$), then $y_i^4 = t_i^2$ (because by assumption $y_i^4 \neq 1$ by Claim 4). Because of Claim 3 we then get that $x^4 = 1$. Moreover $Z(\langle x, y_i \rangle) = \langle x^2, t_i^2, t_i y_i^2 \rangle$ (see Lemma 5.3) and so $Z(\langle x, y_i \rangle)^2 = 1$ for every i such that $t_i^2 \neq 1$.

We claim that $y_i^2 \in \langle t_i \rangle$ and $t_i^2 = 1$ for every $i \ge 2$. Clearly the image $\overline{y_i}$ of y_i in $H = F_i / \langle t_i \rangle$ is central. Because $\langle t_i \rangle \cap \langle t_1 \rangle = 1$ and $y_1^4 = t_1^2$, Lemma 5.5 is applicable to the group H. Indeed, $H = \langle x_1 = \overline{x}, x_2 = \overline{y_1}, x_3 = \overline{y_i} \rangle$, $x_3 \in Z(H)$, $(x_1, t) = x_2^4 = t^2 \neq 1$ and $(x_2, t) = 1$, where $t = (x_2, x_1) = \overline{t_1}$, because $t_1^2 \notin \langle t_i \rangle$. Therefore we get that $x_3^2 \in Z(\langle x_1, x_2 \rangle)^2 = 1$, or equivalently $y_i^2 \in \langle t_i \rangle$, as desired. Assume now that $t_i^2 \neq 1$. Then, by the previous paragraph, $y_i^4 = t_i^2$ and hence $y_i^2 = t_i$, (because the option $y_i^2 = t_i^{-1}$ is excluded by Claim 4). The last part of Claim 4 now implies $x^2 = t_i^2$. Interchanging the role of y_1 and y_i in the above reasoning, we get that $t_1^2 = x^2$. Hence $t_1^2 = t_i^2 \neq 1$, contradicting with $\langle t_1 \rangle \cap \langle t_i \rangle = 1$. This proves that $t_i^2 = 1$ and shows the claim.

Let $i \geq 2$. The natural image of $t_1 y_i$ is central in the non-abelian Kleinian group $F_i/\langle t_1^2 t_i\rangle$. Hence applying Lemma 5.5 to this group, we obtain that $\overline{t_1^2 y_i^2} \in Z(\langle \overline{x}, \overline{y_1}\rangle)^2 = 1$. Consequently $t_1^2 y_i^2 \in \langle t_1^2 t_i\rangle$. Thus $y_i^2 \in \{1, t_i\} \cap \{t_1^2, t_i\} = \{t_i\}$, i.e. $y_i^2 = t_i$. Moreover, Claim 4 implies that either $x^2 = t_1^2$ or $t_1 y_1^2 = x^2$. In the first case, G is a quotient of $\mathcal{T}_{3n}$. In the second case, setting $x' = y_1 x$ and $y_1' = y_1$, one has $t_1' = (y_1, y_1 x) = t_1$ and $t_i' = (y_i, y_1 x) = t_i$. So $y_i^2 = t_i'$ for every $i \geq 2$ and $x'^2 = y_1 x y_1 x = t_1 x y_1 t_1 x y_1 = t_1 x t_1 y_1 x y_1 = x y_1 x y_1 = x t_1 x y_1^2 = t_1^3 x^2 y_1^2 = t_1^3 t_1 y_1^2 y_1^2 = y_1^4 = t_1^2 = (t_1')^2$. This implies that again G is a quotient of $\mathcal{T}_{3n}$. It also finishes the proof of (D) implies (F) for nilpotent groups. ∎

6 (D) implies (F), for non-nilpotent groups

In this section we prove that (D) implies (F) for finite groups that are not nilpotent.

Let G be a finite non-nilpotent group of Kleinian type. By statement 4 of Lemma 5.1, G is a semidirect product $G_3 \rtimes G_2$ of an elementary abelian 3-group G_3 and a 2-group G_2. Moreover, since by assumption G is not nilpotent, statement 1 of Lemma 5.1 implies that G has an abelian subgroup $G_3 \times N_2$ such that N_2 has index 2 in G_2. Thus $G_2 = N_2 \cup N_2 x$, for every $x \in G_2 \setminus N_2$. Let $K = G_3 \cap Z(G)$. Then $G_3 = K \times M$, for some subgroup M of G_3. Note that M is not trivial because, by assumption, G is not nilpotent. For every $m \in M$, let $k_m = m m^x$ and $\widetilde{m} = k_m m$. Since x^2 centralizes G_3 and G_3 is abelian, $k_m \in K$ and thus $\widetilde{m}^x = k_m m^x = k_m^2 m^{-1} = \widetilde{m}^{-1}$. Furthermore $k_{m_1 m_2} = m_1 m_2 (m_1 m_2)^x = k_{m_1} k_{m_2}$ and hence $\widetilde{m_1 m_2} = k_{m_1 m_2} m_1 m_2 = k_{m_1} m_1 k_{m_2} m_2 = \widetilde{m_1}\widetilde{m_2}$. Hence $\widetilde{M} = \{\widetilde{m} \mid m \in M\}$ also is an elementary abelian 3-group and $G_3 = K \times \widetilde{M}$. So, replacing M by $\widetilde{M}$ we may assume that $\widetilde{M} = M$, $a^x = a$ if $a \in K$, and $a^x = a^{-1}$ if $a \in M$. Consequently, $G = K \times (M \rtimes G_2) = K \times (M \rtimes \langle N_2, x\rangle)$, where K and M are elementary abelian 3-groups, $G_2 = \langle N_2, x\rangle = N_2 \cup N_2 x$ is a 2-group, $\langle N_2, M\rangle = N_2 \times M$ is abelian and x acts on M by inversion. Notice that this group is completely determined by N_2, G_2 and the ranks k and m of K and M respectively. To emphasize this information we use the following notation

$$\begin{gathered} G = G_{k,m,N_2,G_2} = K \times (M \rtimes G_2) = K \times ((M \times N_2) : \langle \overline{u}\rangle_2), \\ (k \geq 0, m \geq 1, K = C_3^k, M = C_3^m, u^2 \in N_2 \text{ and } w^u = w^{-1} \text{ for } w \in M) \end{gathered} \tag{8}$$

Since Theorem 1 has already been proved for nilpotent groups, if G is of Kleinian type and G_2 is non-abelian then $K \times G_2$ satisfies either condition (F.1), (F.2) or (F.3). In particular, if $K \neq 1$ and G_2 is non-abelian, then $K \times G_2$ satisfies condition (F.1) and so the exponent of G_2 is 4, $G_2' \subseteq Z(G_2)$ and the exponent of $Z(G_2)$ is 2. In the following four lemmas we find more restrictions on k, m, N_2 and G_2.

Lemma 6.1 *If $G = G_{k,m,N_2,G_2}$ is of Kleinian type then the exponent of G_2 divides 8. Furthermore, if $k \neq 0$ then the exponent of G_2 divides 4 and the exponent of $G_2 \cap Z(G)$ is 1 or 2.*

Proof. First we prove that the exponent of G_2 divides 8. This is a consequence of part 1 of Lemma 5.2, if G_2 is non-abelian. If G_2 is abelian and $g \in G_2$ then $g^2 \in N_2$ and therefore it is central. By statement 5 of Lemma 5.1 the order of g^2 divides 4 and thus the order of g divides 8.

Assume now that $k \neq 0$, or equivalently $K \neq 1$. If G_2 is non-abelian then G_2 has exponent 4 and $Z(G_2)$ has exponent 2 and so $G_2 \cap Z(G)$ has exponent 1 or 2 as wanted. Otherwise, that is if G_2 is abelian, $N_2 = G_2 \cap Z(G)$. Since K has a central element of order 3, the exponent of $Z(G)$ is either 3 or 6 and therefore the exponent of $G_2 \cap Z(G)$ is either 1 or 2. Furthermore $g^2 \in G_2 \cap Z(G)$ for every $g \in G_2$ and so $g^4 = 1$. ∎

Notice that if M_1 is a maximal subgroup of M then $G/(K \times M_1) \cong G_{0,1,N_2,G_2}$. So to obtain restrictions on G_2 and N_2 one may assume without loss of generality that $k = 0$ and $m = 1$. This will be used in the proof of the next three lemmas.

Lemma 6.2 *Assume that $G = G_{k,m,N_2,G_2}$ is of Kleinian type. If L is a normal subgroup of G_2 contained in N_2 such that $G_2/L \cong D_8$ and $a \in N_2$ then $a^2 \in L$. In particular, if $G_2 = D_8$ and a is an element of order 4 in G_2 then $a \notin N_2$.*

Proof. One may suppose that $k = 0$ and $m = 1$. Then $G/L \cong C_3 \rtimes D_8$. If $a^2 \notin L$ and $a \in N_2$ then $G/L = \langle c \rangle_3 \rtimes (\langle \overline{a} \rangle_4 \rtimes \langle b \rangle_2) = \langle c\overline{a} \rangle_{12} \rtimes \langle b \rangle_2 \cong D_{24}$, contradicting statement 3 of Lemma 5.1. ∎

Lemma 6.3 *Assume that $G = G_{k,m,N_2,G_2}$ is of Kleinian type. Let L be a normal subgroup of G_2 contained in N_2. Then G_2/L is not isomorphic to any of the groups Q_{16}, D_{16}^-, D_{16}^+, $\mathcal{D}$, $\mathcal{D}^+$.*

In particular, if, moreover, G_2/L is non-abelian and has order 16 then G_2/L has exponent 4 and the exponent of $Z(G_2/L)$ is 2.

Proof. We may assume that $k = 0$ and $m = 1$ and hence $H = G/L = G_{0,1,Q,P} = \langle w \rangle_3 \rtimes P$, where $P = G_2/L$ and $Q = N_2/L$.

First assume that $P = Q_{16} = \langle a, b \mid a^8 = b^2a^4 = 1, ba = a^{-1}b \rangle$ or $P = D_{16}^- = \langle a, b \mid a^8 = b^2 = 1, ba = a^3b \rangle$. Then $P/\langle a^4 \rangle \cong D_8$ and $a^2 \notin \langle a^4 \rangle$. By Lemma 6.2, $a \notin Q$. However $(\langle w, a^2 \rangle, 1)$ is a strong Shoda pair of H and $[H : \langle w, a^2 \rangle] = 4$, contradicting statement 2(a) of Lemma 5.1.

Second assume that $P = D_{16}^+ = \langle a, b \mid a^8 = b^2 = 1,\ ba = a^5b \rangle$. If $b \in Q$ then $a \notin Q$ and $(A = \langle w, a^2, b \rangle, B = \langle b \rangle)$ is a strong Shoda pair of G with $[A : B] = 12$ and B is not normal in G, contradicting statement 2(b) of Lemma 5.1. On the other hand, if $b \notin Q$ then, interchanging generators if needed, we may assume that $a \in Q$ and hence $(A = \langle w, a \rangle, 1)$ is a strong Shoda pair of H. Let $e = e(H, A, 1)$, a primitive central idempotent of $\mathbb{Q}H$. Then $b^2e = e$ but $be \neq \pm e$ (because be cannot be central in $\mathbb{Q}He$). Hence $\mathbb{Q}He$ is split. Since $|A| = 24$ we obtain a contradiction with statement 2(c) of Lemma 5.1.

Third, assume that $P = \mathcal{D} = \langle a, b, c \mid ca = ac,\ cb = bc,\ a^2 = b^2 = c^4 = 1,\ ba = c^2ab \rangle$. Since ab is of order 4 and $\langle a, b \rangle = D_8$, Lemma 6.2 implies that $ab \notin Q$. It thus follows that either $a \notin Q$ or $b \notin Q$. By symmetry, we may assume that $a \notin Q$ and $b \in Q$. If $c \in Q$ then $(A = \langle w, c, b \rangle, B = \langle b \rangle)$ is a strong Shoda pair of H, $[A : B] = 12$ and $\langle b \rangle$ is not normal in H, contradicting statement 2(b) of Lemma 5.1. If $c \notin Q$ then $(A = \langle w, c^2, b \rangle, B = \langle b \rangle)$ is a strong Shoda pair of H such that $[H : A] = 4$, again in contradiction with statement 2(b) of Lemma 5.1.

Fourth, assume that $P = \mathcal{D}^+ = \langle a, b, c \mid ca = ac,\ cb = bc,\ a^4 = b^2 = c^4 = 1,\ ba = ca^3b \rangle$. Then $a^2c \in P' \subseteq Q$ and $a^2 \in Q$. Thus $c \in Q$. Moreover $P/\langle c \rangle \cong D_8$. By Lemma 6.2, $a \notin Q$. So $(A = \langle M, a^2, c \rangle, B = \langle a^2 \rangle)$ is a strong Shoda pair of H with $[H : A] = 4$. This again yields a contradiction with statement 2(b) of Lemma 5.1.

Now we prove the second statement. Assume that P is non-abelian and of order 16. By statement 3 of Lemma 5.1, P is not isomorphic to D_{16}. By the first part of the lemma, P is not isomorphic to any of the groups: Q_{16}, D_{16}^-, D_{16}^+, D_{16}^-, $\mathcal{D}$. The well known description of the non-abelian groups of order 16 yields that P is isomorphic to one of the groups: $Q_8 \times C_2$, $D_8 \times C_2$, $\mathcal{W}_{21}$ or $\langle a, b \mid a^4 = b^4 = (ab)^2 = (a^2, b) = 1 \rangle$. Hence the result follows. ∎

Lemma 6.4 *Let $G = G_{k,m,N_2,G_2}$ be a finite group of Kleinian type. If G_2 is non-abelian then its exponent is 4, $G_2' \subseteq Z(G_2)$ and $Z(G_2)$ has exponent 2. In particular, Q_{16}, D_{16}^+, D_{16}^-, $\mathcal{D}$ and $\mathcal{D}^+$ are not quotients of G_2.*

Proof. Again we may assume that $G = G_{0,1,N_2,G_2} = \langle w\rangle_3 \rtimes G_2$.

Claim 1. Let $x, y \in G_2$ with $t = (y, x) \neq 1$ and x of order 8. Then t has order 4.

In order to prove this we may assume that $G_2 = \langle x, y\rangle$ and argue by contradiction. So, suppose that t does not have order 4. By statement 6 of Lemma 5.1 and statement 2 of Lemma 5.2, $t \in Z(G)$ and t has order 2. Let $\mathcal{V} = (\langle s\rangle_2 \times \langle y_1^2\rangle_4 \times \langle y_2^2\rangle_4) : (\langle\overline{y_1}\rangle_2 \times \langle\overline{y_2}\rangle_2)$, with $s = (y_2, y_1)$ and $Z(\mathcal{V}) = \langle s, y_1^2, y_2^2\rangle$ (this is the same group $\mathcal{V}$ of Theorem 1 with generators renamed to avoid confusions with the elements t, x and y of G). Then, there is an epimorphism $\mathcal{V} \to G_2$ mapping y_1 to x and y_2 to y. Since $\mathcal{V}/\langle y_2^2, sy_1^4\rangle$ has order 16 and exponent 8, $G_2/\langle y^2, tx^4\rangle$ has order at most 16. However, if $|G_2/\langle y^2, tx^4\rangle| = 16$ then $G_2/\langle y^2, tx^4\rangle \cong \mathcal{V}/\langle y_2^2, sy_1^4\rangle$ and hence $G_2/\langle y^2, tx^4\rangle$ has exponent 8, contradicting Lemma 6.3. This implies that $G_2/\langle y^2\rangle$ has order at most 16 and $G_2/\langle tx^4\rangle$ has order at most 32. Since the latter is non-abelian of exponent 8, it has order 32, by Lemma 6.3. This implies that $y^2 \notin \langle x, t\rangle$. Indeed, for otherwise $|G_2| \leq 32$ and hence $|G_2| = |G_2/\langle tx^4\rangle| = 32$. So $t = x^4$ and therefore $y^2 \in \langle x\rangle$. Thus $|G_2| = 16$, a contradiction. We thus obtain that $G_2/\langle y^2, tx^4\rangle$ has order 8 because we have seen that this group has order at most 8 and $G_2/\langle tx^4\rangle$ has order 32. Moreover, since $|G_2/\langle y^2\rangle| \leq 16$, using again Lemma 6.3, the group $G_2/\langle y^2\rangle$ is either abelian or has exponent 4 and thus either $t \in \langle y^2\rangle$ or $x^4 \in \langle y^2\rangle$. Since $y^2 \notin \langle t, x\rangle$, either $t = y^4$ or $x^4 = y^4$. So in both cases we get $x^2y^2 \notin \langle tx^4\rangle$ and $x^4y^4 \in \langle tx^4\rangle$. This implies that $G_2/\langle tx^4, x^2y^2\rangle$ has order 16 and exponent 8, because $x^4 \notin \langle tx^4\rangle \cup \langle tx^4\rangle x^2y^2 = \langle tx^4, x^2y^2\rangle$. Lemma 6.3 therefore yields that $G_2/\langle tx^4, x^2y^2\rangle$ is abelian, that is, $t \in \langle tx^4, x^2y^2\rangle$. Since $t \notin \langle tx^4\rangle$ we conclude that $y^2 \in \langle x, t\rangle$ a contradiction. This proves the claim.

Claim 2. If $x \in G_2$ has order 8 then $(x, (x, G_2)) = 1$.

It is sufficient to show that if $y \in G_2$ and $t = (y, x) \neq 1$ then $(x, t) = 1$. Assume the contrary, then by Lemma 5.3, $(x, t) = t^2 \neq 1$. Hence Claim 1 implies that both t and t^2 have order 4, a contradiction. This proves Claim 2.

We now first prove by contradiction that G_2 has exponent 4. So assume $x \in G_2$ has order 8. Because of statement 5 of Lemma 5.1, we know that $x \notin Z(G_2)$. Let $y \in G_2$ be so that $t = (y, x) \neq 1$. As before, we may assume that $G_2 = \langle x, y\rangle$. Because of Claim 1, t has order 4 and by the second claim $(x, t) = 1$. By statement 6 of Lemma 5.1, $\langle t^2\rangle$ is a normal subgroup of G contained in N_2. Then, applying Claim 1 to $G/\langle t^2\rangle = G_{0,1,N_2/\langle t^2\rangle,G_2/\langle t^2\rangle}$, we get that $x^4 \in \langle t^2\rangle$. This implies that $t^2 = x^4$. Since t is not central in $\langle x, y\rangle$ (as t has order 4), we get that $(y, t) \neq 1$ and $(yx, t) = (yx, (yx, x)) \neq 1$. Because of Claim 2 we obtain that $y^4 = (yx)^4 = 1$. Moreover, by part 5(a) of Lemma 5.2, $(x^2, y) = t^2$ and, by part 5(b) of the same lemma, $(y^2, x) = 1$. This implies that $y^2, tx^2 \in Z(G_2)$. Since $t \notin Z(G_2)$, we thus have that $G_2/\langle y^2, tx^2\rangle$ is a non-abelian quotient of D_{16}. Since D_{16} is not of Kleinian type, $G_2/\langle y^2, tx^2\rangle$ has order 8 and, from $t^2 = x^4$ and $y^4 = 1$, we have that G_2 has order at most 32. By Lemma 6.3, it follows that G_2 has order exactly 32. Therefore $\langle y^2, tx^2\rangle$ has order 4 and thus $\langle y^2\rangle \cap \langle tx^2\rangle = 1$. Hence, both $G_2/\langle y^2\rangle$ and $G_2/\langle tx^2\rangle$ are non-abelian groups of order 16. Therefore, by Lemma 6.3, both have exponent 4. Thus $x^4 \in \langle y^2\rangle \cap \langle tx^2\rangle = 1$, a contradiction. This finishes the proof of the fact that the exponent of G is 4.

We now prove that $G_2' \subseteq Z(G_2)$. We argue by contradiction. So, because of statement 6 in Lemma 5.1, there exist $x, y \in G_2$ such that $t = (y, x)$ has order 4. One may assume without loss of generality that x and y satisfy condition 1 of Lemma 5.3 (see Remark 5.4), that is $(x, t) = (y, t) = t^2$ and $x^2, y^2 \in Z(G)$. Then $1 = (xy)^4 = xtxy^2xtxy^2 = t^2$, a contradiction.

It remains to show that $Z(G_2)$ has exponent 2. By means of contradiction assume that there exists $z \in Z(G_2)$ of order 4. Since G_2 is not abelian, there exist $x, y \in G_2$ with $(x, y) = t \neq 1$. As before, one may assume that $G_2 = \langle x, y, z\rangle$. Since t has order 2 and z has order 4, $H = \langle t, z^2, x^2, y^2\rangle$ is an elementary abelian 2-subgroup of $Z(G)$ and so there is a subgroup L of index 2 in H which

contains tz^2 but does not contain t. We will use the bar notation for the natural images of the elements of G in G/L. If $x^2 \in L$ we set $x_1 = \overline{x}$, and otherwise we put $x_1 = \overline{tx}$. Similarly, define $y_1 = \overline{y}$ if $y^2 \in L$, and $y_1 = \overline{ty}$ otherwise. Then $G_2/L = \langle x_1, y_1, \overline{z}\rangle$ is a non-abelian epimorphic image of $\mathcal{D}$ with a central element $\overline{z}$ of order 4. This yields a contradiction with Lemma 6.3, because $L \subseteq N_2$. This finishes the proof. ∎

We are ready to finish the proof of Theorem 1 by proving that if G is a non-nilpotent group of Kleinian type then G satisfies condition (F.4). Recall that $G = G_{k,m,N_2,G_2}$ as in (8). Of course, G_2 may be abelian or non-abelian.

Assume first that G_2 is abelian. Then $Z(G) = K \times N_2$. Let u be an element of minimal order in $G_2 \setminus N_2$. Then $G_2 = L \times \langle u\rangle$ and $N_2 = L \times \langle u^2\rangle$. Because of Lemma 6.1, the exponent of G_2 divides 8 and, by statement 5 of Lemma 5.1, the exponent of $Z(G)$ divides 4 or 6. We separately deal with the cases $K = 1$ and $K \neq 1$. Assume that $K = 1$. Then $G = L \times (M \rtimes \langle u\rangle)$ and the exponent of L divides 4. Therefore G is an epimorphic image of $A \times H$, where A is abelian of exponent 4 and H satisfies the first condition of (F.4). Assume now that $K \neq 1$, then the exponent of $Z(G)$ divides 6 and thus the order n of u divides 4. Thus G is an epimorphic image of $A \times H_1$, with A abelian of exponent 6 and $H_1 = M \rtimes C_n = G_{0,m,\langle u^2\rangle,\langle u\rangle_n}$. Then H_1 is an epimorphic image of $H = M \rtimes \mathcal{W}_{11} = G_{0,m,\langle y_1,t,u^2\rangle,\mathcal{W}_{11}}$. We conclude that G is an epimorphic image of $A \times H$, where A and H satisfy the second condition of (F.4).

Now suppose that G_2 is not abelian. Notice that $Z(G_2) \subset N_2$ because N_2 is abelian and $[G_2 : N_2] = 2$. By Lemma 6.4, G_2 has exponent 4 and G_2' has exponent 2. Furthermore, if T is a proper subgroup of G_2' then $G/T \cong G_{k,m,N_2/T,G_2/T}$ and hence, again by Lemma 6.4, the exponent of $Z(G_2/T)$ is 2. It thus follows from Lemma 5.7 that G_2 is an epimorphic image of either $C_2^n \times \mathcal{W}$, $\mathcal{W}_{1n}$ or $\mathcal{W}_{2n}$ for some n.

Assume that G_2 is an epimorphic image of $C_2^n \times \mathcal{W}$, but not of $\mathcal{W}_{in}$ for $i = 1, 2$ and some n. This implies that $G_2 = C_2^r \times \mathcal{W}$ for some r. Having in mind that $Z(G_2) \subseteq N_2$ one has that $G = A \times G_{0,m,Q,\mathcal{W}}$ for A an abelian group of exponent dividing 6 and Q and abelian subgroup of index 2 in $\mathcal{W}$. Let $L_1 = \langle x^2, y^2\rangle, L_2 = \langle x^2, (xy)^2\rangle$ and $L_3 = \langle y^2, (xy)^2\rangle$. Then $L_i \subseteq Q$ and $\mathcal{W}/L_i \cong D_8$. Further, the image of xy (resp. y, x) in $\mathcal{W}/L_1$ (resp. $\mathcal{W}/L_2$, $\mathcal{W}/L_3$) has order 4. Thus, $xy, x, y \notin Q$, contradicting the fact the $[\mathcal{W} : Q] = 2$.

In the remainder of the proof we assume that G_2 is an epimorphic image of $\mathcal{W}_{1n}$ or $\mathcal{W}_{2n}$. For simplicity, the symbols used for the generators in the description of the groups $\mathcal{W}_{1n}$ and $\mathcal{W}_{2n}$, as given in part (F) of Theorem 1, also will be used for their images in G_2. So we write $G_2 = \langle x, y_1, \cdots, y_n, t_1 = (y_1, x), \ldots, t_n = (y_n, x)\rangle$ with the respective relations. Then G_2' is an elementary abelian 2-group and $G_2' = \langle t_1, \ldots, t_n\rangle$. Assume that $|G_2'| = 2^r$. Then, reordering the y_i's, one may assume that $G_2' = \langle t_1, \ldots, t_r\rangle$. Let $r < i \leq n$ and let $t_i = t_1^{\alpha_1} \cdots t_r^{\alpha_r}$, with $\alpha_i = 0$ or 1. Then $y_i' = y_i y_1^{\alpha_1} \cdots y_r^{\alpha_r} \in Z(G_2) \subseteq N_2$, for $i > r$. Thus, replacing y_i by y_i' for $i > r$ one has that $G_2 = B \times P$, where $B = \langle y_{r+1}, \ldots, y_n\rangle$, an elementary abelian 2-group, and P an epimorphic image of W_{1r} or W_{2r} such that $P' = \langle t_1\rangle_2 \times \cdots \times \langle t_r\rangle_2$. Then the map $f : P' \to \langle y_1, \ldots, y_r\rangle$ given by $f(t_1^{\alpha_1} \cdots t_r^{\alpha_r}) = y_1^{\alpha_1} \cdots y_r^{\alpha_r}$ ($\alpha_i = 0$ or 1) is well defined. Moreover $(x, f(s)) = s$ and therefore $(xf(s))^2 = sx^2 f(s)^2$, for every $s \in P'$.

Assume that P is a quotient of $\mathcal{W}_{1r}$. Let $A_1 = K \times B$, an abelian group of exponent dividing 6. Then $G = A_1 \times H_1$, where $H_1 = G_{0,m,Q,P}$ and Q is an abelian subgroup of index 2 in P. We will show that G is an epimorphic image of $A \times H$, with A and H satisfying the second condition of $(F.4)$. For this it is enough to show that one may assume that $y_1, \ldots, y_r, t_1, \ldots, t_r, x^2 \in Q$. Obviously $t_1, \ldots, t_r, x^2 \in Q$. Assume that $y_{i_0} \notin Q$. We separately deal with the cases $x^2 \in \langle t_{i_0}\rangle$ and $x^2 \notin \langle t_{i_0}\rangle$. If $x^2 \notin \langle t_{i_0}\rangle$ then $K_1 = \langle x, y_{i_0}\rangle/\langle x^2\rangle$ and $K_2 = \langle x, y_{i_0}\rangle/\langle x^2 t_{i_0}\rangle$ are isomorphic to D_8. Moreover $\overline{x}$ has order 4 in K_2 and $\overline{xy_{i_0}}$ has order 4 in K_1. Hence $x, xy_{i_0} \notin Q$, by Lemma 6.2,

and $y_{i_0} \notin Q$, yielding a contradiction. Suppose now that $x^2 \in \langle t_{i_0}\rangle$. Then, replacing x by xt_{i_0}, if needed, one may assume that $x^2 = 1$. Then, for every $i = 1, \dots, r$, we have $\langle x, y_i\rangle \cong D_8$ and xy_i has order 4. Therefore $xy_i \notin Q$, by Lemma 6.2. Since $y_{i_0} \notin Q$, one gets that $x \in Q$. For every $1 \neq y \in \langle y_1, \dots, y_r\rangle$, the group $\langle x, y\rangle$ is not abelian. Then $y \notin Q$. That is $\langle y_1, \dots, y_r\rangle \cap Q = 1$ and hence $r = 1$. Thus $P = \langle x, y_1\rangle \cong D_8$, with $x^2 = 1 = y_i^2$ and $Q = \langle x, t\rangle$. Interchanging the roles of x and y_1 one may assume that $y_1 \in Q$ as desired.

Finally, assume that P is a quotient of $\mathcal{W}_{2r}$, with $|P'| = 2^r$. Hence, $f(s)^2 = s$ for every $s \in P'$. We also assume that P is not an epimorphic image of $\mathcal{W}_{1h}$ for any $h \geq 1$. We claim that if $r = 1$ then the exponent of A_1 divides 2. Notice that P is non-abelian, $\mathcal{W}_{21}$ has order 16 and D_8 is an epimorphic image of $\mathcal{W}_{11}$. Then P is isomorphic to either $\mathcal{W}_{21}$ or Q_8. This implies that $K \times H_2$ is an epimorphic image of G, where $H_2 = G_{0,1,\langle a\rangle,Q_8}$ and a is an element of order 4 in Q_8. Then H_2 has a cyclic subgroup K_2 of index 2 and so $(K_2, 1)$ is a strong Shoda pair of $H_2 = G_{0,1,\langle a\rangle,Q_8}$. Thus $e = e(H_2, K_2, 1)$ is a primitive central idempotent of $\mathbb{Q}H_2$ and, applying Proposition 4.1, one has $\mathbb{Q}H_2e \cong \mathbb{H}(\mathbb{Q}(\sqrt{3}))$. Therefore, if $K \neq 1$ then $\mathbb{Q}G$ has a quotient isomorphic to $\mathbb{Q}(\xi_3) \otimes_{\mathbb{Q}} \mathbb{H}(\mathbb{Q}(\sqrt{3})) \simeq M_2(\xi_3, \sqrt{3})$, contradicting statement 2 of Lemma 5.1. This proves the claim.

Now we separately deal with the cases $x^2 \notin P' \setminus \{1\}$ and $x^2 \in P' \setminus \{1\}$. In both cases we will show that $r = 1$ and hence, by the above, $K = 1$ and $G = B \times H_1$, where B is an elementary abelian 2-group and $H_1 = G_{0,m,Q,P}$ with Q is an abelian subgroup of index 2 in P. Then, in order to show that G is an epimorphic image of $A \times H$ with A and H satisfying the third condition of (F.4), it is enough to prove that one may assume that $x \in Q$. Suppose $x^2 \notin P' \setminus \{1\}$. Then $\langle x, f(s)\rangle/\langle x^2\rangle$ is isomorphic to D_8, for every $s \in P' \setminus \{1\}$. By Lemma 6.2, $f(s) \notin Q$ because the natural image of $f(s)$ in $\langle x, f(s)\rangle/\langle x^2\rangle$ has order 4. This implies that $r = 1$, because otherwise $y_1, y_2, y_1y_2 \notin Q$, contradicting the fact that $[P : Q] = 2$. Then, replacing x by xy_1 if needed, one may assume that $x \in Q$. Assume now that $x^2 \in P' \setminus \{1\}$. We claim that one may assume that $x^2 = t_1$. If $x^2 \notin \langle t_2, \dots, t_r\rangle$ this is obtained by replacing y_1 by $f(x^2)$. Otherwise $x^2 = t_2^{\alpha_2} \cdots t_r^{\alpha_r}$, for some $\alpha_1, \dots, \alpha_r \in \{0, 1\}$ with $\alpha_i = 1$ for some i. Then, replacing $\{y_1, \dots, y_r\}$ by $\{f(x^2), y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_r\}$, one obtains the desired conclusion. So we assume that $x^2 = t_1$. Then $\overline{f(s)}$ has order 4 in $\langle x, f(s)\rangle/\langle x^2\rangle \simeq D_8$ for every $s \in P' \setminus \langle t_1\rangle$ and therefore $(P' \setminus \langle t_1\rangle) \cap Q = 1$. This implies that $r \leq 2$. If $r = 2$ then $y_1y_2, y_2 \notin Q$ and therefore $y_1 \in Q$. Replacing x by xy_2 if needed, one may assume that $x \in Q$. So $Q = \langle x, y_1, y_2^2\rangle$. Let $1 \neq m \in M$. Then $(U = \langle m, x, y_2^2\rangle, \langle y_2^2\rangle)$ is a strong Shoda pair of $H = \langle m, P\rangle$ and $[H : U] = 4$, contradicting statement 2 of Lemma 5.1. Thus $r = 1$ and $x^2 = t_1 = y_1^2$. Therefore $P \cong Q_8$ and either x or y_1 does not belong to Q. By symmetry, one may assume that $y_1 \notin Q$ and, replacing x by xy_1 if needed, one may assume that $x \in Q$. This finishes the proof of Theorem 1.

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